Strategic Knowledge Acquisition

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The article proposes a trimodal logical system that can express the strategic ability of coalitions to learn from their experience. The main technical result is the completeness of the proposed system.

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1 INTRODUCTION

In this article we study the knowledge acquired by agents while interacting with each other. For example, consider a situation in which nurses n_1 and n_2 are in charge of patients p_1 and p_2, respectively, during a deadly virus epidemic. The nurses are being supervised by a doctor d. It is known that each patient will die the next day unless the patient is given a medicine that can cure the disease. The situation is complicated by the fact that there are several different strains of the virus against which there are three different drugs a, b, and c. If a wrong drug is given, the patient dies the next day. To keep our example simple, we assume that each nurse must administer exactly one drug.

We capture this setting through the game depicted in Figure 1. This game has three "initial" states: q_1, q_2, and q_3 and four "final states": q_4, q_5, q_6, and q_7. Variables v_a, v_b, and v_c represent the drugs effective in the initial states. Variables $alive_1$ and $alive_2$ show which of the two patients is alive in which of the final states. The directed edges on the diagram represent the possible transitions between states. For example, if in state q_1 drug a (which is effective in this state) is given to the first patient and drug b is given to the second patient, then the system transitions into state q_4 in which the first patient is alive and the second is not. This is captured by variable v_a being true in state q_1, by label ab on the directed edge from state q_1 to state q_4, and by variable $alive_1$ being true in state q_4. In this example we assume that the doctor's action does not influence the outcome.

Note that if in state q_1 nurse n_1 uses drug a, then patient p_1 will be alive the next day no matter what the action of the nurse n_2 is. We write this as $q_1 \Vdash [(n_1, a)]alive_1$. If both nurses use drug a in state q_1, then both patients will be alive: $q_1 \Vdash [(n_1, a), (n_2, a)](alive_1 \wedge alive_2)$.

In addition to explicit strategy modality $[s]\varphi$ that states that statement φ *will be* true after strategy s is executed, we also consider modality $[s]^{-1}\varphi$ that states that φ *was* true before the execution of s. For example, $q_4 \Vdash [(n_1, a)]^{-1}v_a$ means that if nurse n_1 used drug a during the transition to state q_4, then variable v_a was true in the previous state. Note that $[s]^{-1}\varphi$ is *not* the same as dual modality $\neg[s]\neg\varphi$. Modality $[s]^{-1}\varphi$ refers to the *past* state, while modality $\neg[s]\neg\varphi$ refers to the *future* state.

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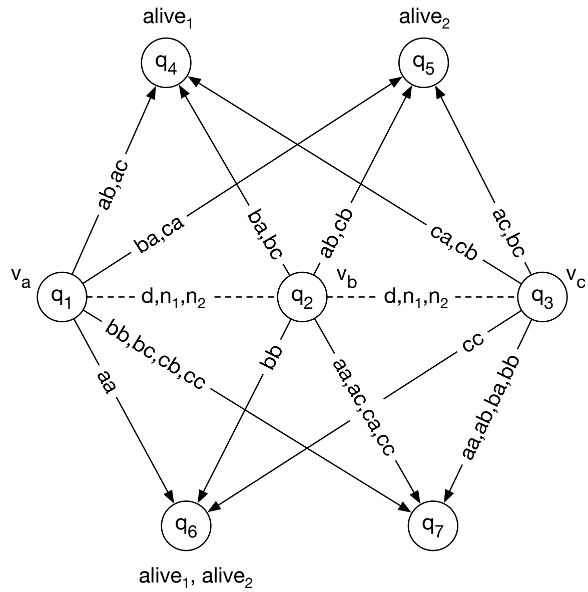

Fig. 1. Epidemic Game.

Imagine now that in state q_1 nurse n_1 uses drug b and nurse n_2 uses drug c. The next day both patients die and, thus, the system transitions from state q_1 to state q_7. Since the state is q_7 and the first nurse gave drug b, then b could not have been the right drug to use: $q_7 \Vdash [(n_1, b)]^{-1}\neg v_b$. Furthermore, under the same assumption that one of the drugs a and c is the cure: $q_7 \Vdash [(n_1, b)]^{-1}(v_a \vee v_c)$. However, since state q_7 could be reached by giving the first patient drug b from either of the two states: q_1 and q_3, there is not enough evidence to conclude that a is the right drug: $q_7 \Vdash \neg[(n_1, b)]^{-1}v_a$. At the same time, if one specifies which drugs were given to both patients, then in state q_7 one might claim that drug a was the correct one: $q_7 \Vdash [(n_1, b)(n_2, c)]^{-1}v_a$. Not only is statement $[(n_1, b)(n_2, c)]^{-1}v_a$ true in state q_7, but this statement is also known to the doctor who examined the patient and, thus, can distinguish state q_7 from the other outcome states: $q_7 \Vdash \mathsf{K}_d[(n_1, b)(n_2, c)]^{-1}v_a$. In fact, the doctor also knows that any of the two patients will be alive if given drug a:

$$q_7 \Vdash \mathsf{K}_d[(n_1, b), (n_2, c)]^{-1} \bigwedge_{i \le 2}[(n_i, a)]alive_i. \tag{1}$$

We assume that a priori the doctor did not know which drug is the correct one. In Figure 1, dashed lines labeled with d, n_1, and n_2 between initial states q_1, q_2, and q_3 show that doctor d and nurses n_1 and n_2 cannot distinguish these states. Thus,

$$q_1 \Vdash \neg\mathsf{K}_d \bigwedge_{i \le 2}[(n_i, a)]alive_i. \tag{2}$$

Together, formula (1) and formula (2) state that the doctor *learned how* the patients could have been saved.

Note that statement (1) is not true in state q_4, where the first patient is alive. However, if the same drugs are given to the patients and the system transitions to state q_4, then the doctor learns that the initial state was q_2 and, thus, that drug b is the cure:

$$q_4 \Vdash \mathsf{K}_d[(n_1,b),(n_2,c)]^{-1}\bigwedge_{i\leq 2}[(n_i,b)]alive_i.$$

In fact, if the patients are given drugs b and c, then the doctor learns, in either of the outcomes, which drug is the cure:

$$q_j \Vdash \bigvee_{z\in\{a,b,c\}}\mathsf{K}_d[(n_1,b),(n_2,c)]^{-1}\bigwedge_{i\leq 2}[(n_i,z)]alive_i,$$

where $4\leq j\leq 7$. Thus, if the patients are given drugs b and c in any of the states q_1, q_2, and q_3, then the doctor will learn how to cure the patients:

$$q_j \Vdash \quad [(n_1,b),(n_2,c)]\bigvee_{z\in\{a,b,c\}}\mathsf{K}_d[(n_1,b),(n_2,c)]^{-1}\bigwedge_{i\leq 2}[(n_i,z)]alive_i,$$

where $1\leq j\leq 3$. Although the doctor cannot distinguish initial states q_1, q_2, and q_3, the doctor *knows* that she will learn this because the above statement is true in all of the three indistinguishable initial states. Hence,

$$q_1 \Vdash \quad \mathsf{K}_d[(n_1,b),(n_2,c)]\bigvee_{z\in\{a,b,c\}}\mathsf{K}_d[(n_1,b),(n_2,c)]^{-1}\bigwedge_{i\leq 2}[(n_i,z)]alive_i. \tag{3}$$

Of course, there is nothing special about drugs b and c. The same is true if any two different drugs are given to the patients and the doctor knows this:

$$q_1 \Vdash \quad \mathsf{K}_d\bigwedge_{x\neq y\in\{a,b,c\}}[(n_1,x),(n_2,y)]\bigvee_{z\in\{a,b,c\}}\mathsf{K}_d[(n_1,x),(n_2,y)]^{-1}\bigwedge_{i\leq 2}[(n_i,z)]alive_i.$$

In other words, the doctor *knows how* to *learn how* she could save future patients from this deadly epidemic.

As we have seen from the above examples, the combination of modalities K, $[s]$, and $[s]^{-1}$, could be used to reason about agents and coalitions abilities not only to *know how* they can achieve a certain result, but also to reason about their ability to *learn how* from a specific experience. In this article we give a sound and complete axiomatic system describing the interplay of these three modalities.

The rest of this article is structured as follows. In the next section we review the related literature. Section 3 introduces the syntax and the formal semantics of our logical system. Section 4 proves that strategic knowledge acquisition cannot be expressed in the language without modality $[s]^{-1}$. Section 5 lists and discusses the axioms and the inference rules of our system. In Section 6 and Section 7 we prove, respectively, the soundness and the strong completeness completeness of our system. Section 8 considers the class of games that can only have actions that are explicitly mentioned in the language. It is shown that no strongly sound logical system can be strongly complete with respect to this class. Section 9 concludes.

2 RELATED LITERATURE

Harel, Kozen, and Tiuryn distinguish two main approaches to modal logics of programs (or, in our case, strategies): *exogenous* and *endogenous* [24, p.157]. Under the exogenous approach the programs are an explicit part of the logical syntax. Under the endogenous one the programs are only implicitly referred to by the syntax. The logical system proposed in this article follows the exogenous approach.

The best known example of the *exogenous* approach is the dynamic logic [24, 37] and its precursor Hoare logic [25]. The completeness theorem for the dynamic logic is proven in [35]. The dynamic logic uses modality $[s]\varphi$, where label s is a program that specifies how the machine will navigate through a sequence of states. De Giacomo and Lenzerini extended dynamic logic with converse modality that, just like our modality $[s]^{-1}$, refers to past transitions [18]. Another example of exogenous approach is Joint Action logic [2]. This logic axiomatizes properties of a joint action modality, which is very similar to our modality $[s]$. Unlike the logic proposed in the current article, Joint Action logic includes neither knowledge modality K_C nor past action modality $[s]^{-1}$.

Half-way between the *exogenous* and the *endogenous* approaches is the strategy logic [17, 30]. This logic combines quantifiers over strategies and temporal modalities. For example, formula $\exists s\,(a,s)\,\mathsf{X}\,\varphi$ in the strategy logic states that there is a strategy s that could be used by agent a to achieve φ on the next step. Belardinelli [8] proposed to extend the syntax of the logic with the individual knowledge modality K_a. For example, formula $\exists s\,\mathsf{K}_a(a,s)\,\mathsf{X}\,\varphi$ means that agent a knows a strategy that she can use to achieve φ on the next step. The literature on the strategy logic covers model checking [10, 13], synthesis [16], decidability [29, 39], and bisimulation [9]. Aminof et al. proposed a probabilistic strategy logic [6]. Note that modality $[(a_1,s_1),\dots,(a_n,s_n)]\,\varphi$ in our logical system corresponds to $(a_1,s_1)\dots(a_n,s_n)\,\mathsf{X}\,\varphi$ in the strategy logic. Thus, the logical system proposed in this article could be viewed as a partial universal fragment of the strategy logic extended with modalities K_C and $[s]^{-1}$. There are no known complete axiomatizations for any of the strategy logics.

An *endogenous* logical system for reasoning about strategic abilities was proposed by Marc Pauly [36]. It uses modality $\mathsf{S}_C\varphi$ that stands for "there is a strategy of coalition C that guarantees φ". His approach has been widely studied by others [4, 21, 22]. Alur, Henzinger, and Kupferman introduced Alternating-Time Temporal Logic (ATL) that combines temporal and coalition modalities [5]. Van der Hoek and Wooldridge proposed to combine ATL with the epistemic modality to form Alternating-Time Temporal Epistemic Logic [38]. Goranko and van Drimmelen gave a complete axiomatization of ATL [23]. Walther, van der Hoek, and Wooldridge combined the exogenous and endogenous approaches in what they called ATL with explicit strategies and gave its complete axiomatization [40]. They system, just like ATL, can express properties of computations paths, while the system in the current article cannot. At the same time, [40] is missing knowledge modality and ability to refer to the past, which are present in our logic. Decidability and model checking problems for ATL-like systems has also been widely studied [7, 12, 13]. An alternative approach to expressing the power to achieve a goal in a temporal setting is the STIT logic [11, 26, 27, 34, 42]. Modality "past" in the context of STIT logic is briefly discussed in [28]. Broersen, Herzig, and Troquard have shown that coalition logic can be embedded into a variation of STIT logic [14]. Ågotnes and Alechina [1] proposed a complete logical system for modalities S_C and K_C.

Note that in case of endogenous logical systems, there is a difference between having a strategy and knowing the strategy. This difference has been explored in recent works on know-how strategies [3, 20, 31–33, 41]. The difference does not exist in exogenous logical systems because the strategy is explicitly mentioned in the modality.

3 SYNTAX AND SEMANTICS

In this article we assume a fixed set of agents $\mathcal{A}$, a fixed set of actions Δ_0, and a fixed set of propositional variables. By a coalition we mean an arbitrary subset of $\mathcal{A}$.

Definition 1. *Language Φ is defined by grammar*

$$\varphi := p \mid \neg\varphi \mid \varphi\to\varphi \mid \mathsf{K}_C\varphi \mid [s]\varphi \mid [s]^{-1}\varphi,$$

where C ranges over coalitions and s over all possible functional relations such that $s\subseteq\mathcal{A}\times\Delta_0$.

We read $\mathsf{K}_C\varphi$ as "coalition C distributively knows φ", $[s]\varphi$ as "statement φ *will be* true after strategy s is executed", and $[s]^{-1}\varphi$ as "statement φ *was* true before the execution of s".

We think that distributed knowledge is an important form of knowledge to consider in the strategic context. For example, if an agent wants to achieve a goal in secrecy from group C, whose members might potentially communicate with each other, then the agent wants to prevent "distributed" learning by group C of the fact that the goal is achieved. The individual knowledge (each member of a group C knows φ) is expressible through distributed knowledge as $\bigwedge_{a\in C}\mathsf{K}_{\{a\}}\varphi$. The other well-known form of group knowledge, common knowledge, has fewer interesting strategy-related properties than distributed knowledge. For example, if coalition C distributively knows a strategy to achieve φ and a disjoint coalition D distributively knows a strategy to achieve ψ, then coalition $C \cup D$ distributively knows a strategy to achieve $\varphi \wedge \psi$. The same is not true for common knowledge.

For any two sets X and Y, let X^Y denote the set of all functional relations (functions) from set Y to set X.

Definition 2. *A game is a tuple* $(W, \{\sim_a\}_{a\in\mathcal{A}}, \Delta, M, \pi)$ *such that*

(1) W *is a set of "states",*
(2) $\sim_a$ *is an "indistinguishability" equivalence relation on set* W *for each agent* $a \in \mathcal{A}$,
(3) Δ *is a set, called "domain of actions", where* $\Delta_0 \subseteq \Delta$,
(4) $M \subseteq W \times \Delta^{\mathcal{A}} \times W$ *is a "mechanism" relation,*
(5) $\pi(p) \subseteq W$ *for each propositional variable* p.

By a *complete action profile* we mean an arbitrary (total) functional relation from set $\Delta^{\mathcal{A}}$. In Section 8 we discuss the reason for distinguishing the set of actions Δ_0 available in the language from the set of actions Δ available in the game.

In our introductory example, see Figure 1, there are three agents: n_1, n_2, and d. The set W consists of states q_1, q_2, q_3, q_4, q_5, q_6, and q_7. None of the agents can distinguish the "initial" states q_1, q_2, and q_3. The set of actions Δ consists of the three drugs: a, b, and c.

Recall that in our introductory example the action of the doctor d does not influence the outcome. As a result, we denote the complete actions profiles by pairs of actions: aa, ab, etc. In general, a complete action profile is a functional relation between agents and actions. Thus, for example, profile ab is the relation $\{(n_1, a), (n_2, b), (d, x)\}$ where x is an element of the set $\{a, b, c\}$. The mechanism M is represented in the figure by labeled directed edges. For example, label ab on directed edge from state q_1 to state q_4 means that $(q_1, \{(n_1, a), (n_2, b), (d, x)\}, q_4) \in M$. Note that we define the mechanism as a relation, not a function. Thus, the mechanism can be non-deterministic. It is also possible that for some combination of a state and a complete action profile there might be no next state. Informally, we interpret this as the ability of the players to halt the game.

As described, the epidemic game has explicit initial and final states, but Definition 2 does not distinguish between different types of states. In general, we assume that once the system transitions into a new state, the agents will take actions to transition to yet another state. For example, in the setting of our example, suppose that drug a is the cure and nurse n_1 is expected to get a new patient p_3 the next day. Thus, if nurse n_1 administers drug a to patient p_1 today, then she will know tomorrow that drug a will cure patient p_3 the day after tomorrow:

$$[(n_1, a)]\mathsf{K}_{n_1}[(n_1, a)]alive_3.$$

We conclude this section with the key definition of this article. It specifies the meaning of modalities K_C, $[s]$, and $[s]^{-1}$. We write $w \sim_C u$ if $w \sim_a u$ for each agent $a \in C$. In particular, $w \sim_\varnothing u$ is true for any two states w and u.

Definition 3. *For any game* $(W, \{\sim_a\}_{a\in\mathcal{A}}, \Delta, M, \pi)$*, any state* $w \in W$*, and any formula* $\varphi \in \Phi$*, satisfaction relation* $w \Vdash \varphi$ *is defined as follows:*

(1) $w \Vdash p$ *if* $w \in \pi(p)$*, where* p *is a propositional variable,*
(2) $w \Vdash \neg\varphi$ *if* $w \nVdash \varphi$*,*
(3) $w \Vdash \varphi \to \psi$ *if* $w \nVdash \varphi$ *or* $w \Vdash \psi$*,*
(4) $w \Vdash \mathsf{K}_C\varphi$ *if* $u \Vdash \varphi$ *for each* $u \in W$ *such that* $w \sim_C u$*,*
(5) $w \Vdash [s]\varphi$ *if* $u \Vdash \varphi$ *for each state* $u \in W$ *and each profile* $\delta \in \Delta^{\mathcal{A}}$ *such that* $s \subseteq \delta$ *and* $(w, \delta, u) \in M$*,*
(6) $w \Vdash [s]^{-1}\varphi$ *if* $u \Vdash \varphi$ *for each* $u \in W$ *and each profile* $\delta \in \Delta^{\mathcal{A}}$ *such that* $s \subseteq \delta$ *and* $(u, \delta, w) \in M$*.*

4 UNDEFINABILITY OF STRATEGIC KNOWLEDGE ACQUISITION WITHOUT MODALITY $[]^{-1}$

As we have seen in Section 1, language Φ is suitable for expressing various statements about strategic knowledge acquisition. In this section we show that the presence of modality $[s]^{-1}$ in language Φ is crucial. Namely, we give an example of a statement about strategic knowledge acquisition which is expressible in language Φ but is not expressible in language Φ^-:

$$\varphi := p \mid \neg\varphi \mid \varphi \to \varphi \mid \mathsf{K}_C\varphi \mid [s]\varphi,$$

obtained from language Φ by removing modality $[s]^{-1}$. Note that because modality $[s]^{-1}$ is the only part of our syntax that refers to the previous states, it is relatively easy to show that this modality itself cannot be expressed in language Φ^-. This is *not* what we do in this section. Instead, we construct such formula ζ not expressible in language Φ^-, that meaning of statement $w \Vdash \zeta$ depends only on *the states of the model reachable from state* w and not on *the states from which state* w *is reachable*. Furthermore, formula ζ is a modified version of statement (3) from the introduction:

$$\zeta = \mathsf{K}_n[(n,a)] \bigvee_{x\in\{b,c\}} \mathsf{K}_n[(n,a)]^{-1}[(n,x)]p.$$

Informally, this formula states that nurse n knows that by administering drug a she will find out which of drugs b and c achieves condition p. Without loss of generality, in this section we assume that set $\mathcal{A}$ contains only agent n, set Δ_0 contains only actions a, b, and c, and that the set of propositional variables contains only p.

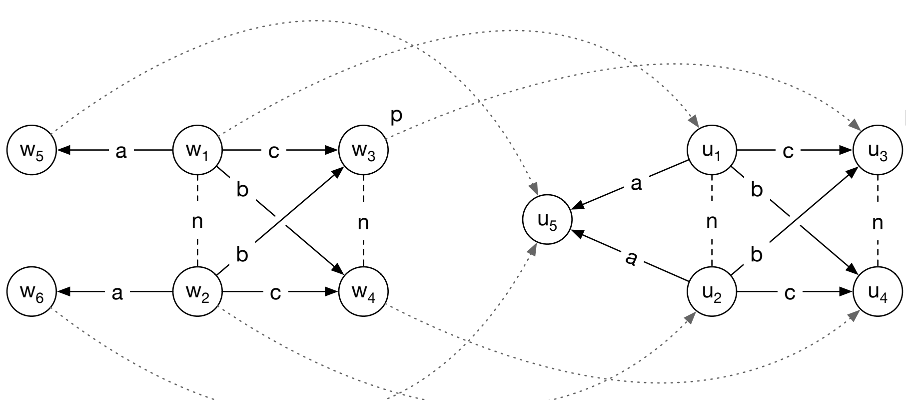

Fig. 2. Homomorphism h between two games is shown using dotted arrows.

We now show that formula ζ is not expressible in language Φ^{-}. To do this, we consider two single-agent games depicted in Figure 2. We refer to them as the left and the right games. In both of these games agent n has three actions: a, b, and c that transition the games between states as shown in the diagram. In the left game nurse n cannot distinguish state w_1 from w_2 and state w_3 from w_4, while in the right game she cannot distinguish state u_1 from u_2 and state u_3 from u_4. Propositional variable p is satisfied only in state w_3 of the left model and state u_3 of the right model. We refer to the satisfaction relations for the left game and the right game as $\Vdash_l$ and $\Vdash_r$ respectively. Similarly, by $\sim_n^l$ and $\sim_n^r$ we mean the indistinguishability relation of agent n in the left and the right games, respectively. Finally, by M_l and M_r we mean the mechanisms of the left and the right games.

We define homomorphism h of these two models as following

$$h(w_i) = \begin{cases} u_i, & \text{if } i \neq 6, \\ u_5, & \text{if } i = 6. \end{cases}$$

Homomorphism h is shown in the diagram using dotted lines. We prove the undefinability of formula ζ in language Φ^{-} in three steps. First, in Lemma 4 we show that these two models are not distinguishable in language Φ^{-}. Then, in Lemma 5 and Lemma 6 we prove that formula ζ is satisfied in one of the states of the left model, but is not satisfied in the corresponding state of the right model. Together, these three lemmas imply undefinability of formula ζ in language Φ^{-}. We formally state this result as Theorem 1.

We start the proof with two technical lemmas that we use in the proof of Lemma 4. The statements of both lemmas follows from the definition of the two models, see Figure 2.

Lemma 1. *$(w, \delta, w') \in M_l$ iff $(h(w), \delta, h(w')) \in M_r$ for any states w, w' of the left game and any action profile δ.* □

Lemma 2. *If $w \sim_n^l w'$, then $h(w) \sim_n^r h(w')$, for any states w, w' of the left game.* □

Lemma 3. *If $h(w) \sim_n^r h(w')$, then $w \sim_n^l w'$, for any states w, w' of the left game such that $w \notin \{w_5, w_6\}$.* □

Lemma 4. *$w \Vdash_l \varphi$ iff $h(w) \Vdash_r \varphi$ for state w of the left model and any formula $\varphi \in \Phi^{-}$.*

Proof. We prove this lemma by induction on structural complexity of formula φ. Recall that propositional variable p is satisfied only in state w_3 of the left game and state $u_3 = h(w_3)$ of the right model. Thus, $w \Vdash_l p$ if and only if $h(w) \Vdash_r p$.

If formula φ is a negation or an implication, then required follows from items (2) and (3) of Definition 3 and the induction hypothesis in the standard way.

Suppose that formula φ has the form $[s]\psi$.

$(\Rightarrow)$: If $h(w) \nVdash_r [s]\psi$, then, by item (5) of Definition 3, there is a state u and an action profile δ such that $s \subseteq \delta$, $(h(w), \delta, u) \in M$, and $u \nVdash_r \psi$. Note that $u = h(w')$ for some state w' of the left model because function h is a surjection, see Figure 2. Thus, $(w, \delta, w') \in M$, and $w' \nVdash \psi$, respectively by Lemma 1 and the induction hypothesis. Therefore, $w \nVdash_l [s]\psi$ by item (5) of Definition 3.

$(\Leftarrow)$: The proof in this direction is similar to the one above except that it does *not* use the fact that function h is a surjection.

Finally, assume that formula φ has the form $\mathsf{K}_n\psi$.

$(\Rightarrow)$: If $h(w) \nVdash_r \mathsf{K}_n\psi$, then, by item (4) of Definition 3, there is a state u of the right game such that $h(w) \sim_n^r u$ and $u \nVdash_r \psi$. We consider the following two cases separately:

Case I: $w \in \{w_5, w_6\}$. Thus, $h(w) = u_5$, see Figure 2. Also, $u = u_5$ because $u_5 = h(w) \sim_n^r u$ see Figure 2. Hence, $u = h(w)$. Then, $h(w) \nVdash_r \psi$ by the assumption $u \nVdash_r \psi$. Thus, $w \nVdash_l \psi$ by the induction hypothesis. Therefore, $w \nVdash_l \mathsf{K}_n\psi$ by item (4) of Definition 3.

Case II: $w \notin \{w_5, w_6\}$. Note that $u = h(w')$ for some state w' of the left model because function h is a surjection, see Figure 2. Thus, $w \sim_n^l w'$ by Lemma 3 and the assumption $w \notin \{w_5, w_6\}$ of the case. Also, $w' \nVdash \psi$ by the induction hypothesis. Therefore, $w \nVdash \mathsf{K}_n\psi$ by item (4) of Definition 3.

$(\Leftarrow)$: If $w \nVdash_r \mathsf{K}_n\psi$, then, by item (4) of Definition 3, there is a state w' of the left game such that $w \sim_n^l w'$ and $w \nVdash_l \psi$. Thus, $h(w) \sim_n^r h(w')$ and $h(w) \nVdash_r \psi$ by Lemma 2 and the induction hypothesis, respectively. Therefore, $h(w) \nVdash_r \mathsf{K}_n\psi$ by item (4) of Definition 3. □

Lemma 5. *$w_1 \Vdash_l \zeta$.*

Proof. Suppose that $w_1 \nVdash_l \mathsf{K}_n[(n,a)] \bigvee_{x\in\{b,c\}} \mathsf{K}_n[(n,a)]^{-1}[(n,x)]p$. Thus, by item (4) of Definition 3, there is a state w' of the left model such that $w_1 \sim_n^l w'$ and

$$w' \nVdash [(n,a)] \bigvee_{x\in\{b,c\}} \mathsf{K}_n[(n,a)]^{-1}[(n,x)]p.$$

Note that statement $w_1 \sim_n^l w'$ implies that either $w' = w_1$ or $w' = w_2$, see Figure 2. We consider these two cases separately:

Case I: $w_1 \nVdash [(n,a)] \bigvee_{x\in\{b,c\}} \mathsf{K}_n[(n,a)]^{-1}[(n,x)]p$. Thus, by item (5) of Definition 3, there is a state w_1' of the left game such that $(w_1, \{(n,a)\}, w_1') \in M_l$ and $w_1' \nVdash_l \bigvee_{x\in\{b,c\}} \mathsf{K}_n[(n,a)]^{-1}[(n,x)]p$. Statement $(w_1, \{(n,a)\}, w_1') \in M_l$ implies that $w_1' = w_5$, see Figure 2. Hence,

$$w_5 \nVdash_l \bigvee_{x\in\{b,c\}} \mathsf{K}_n[(n,a)]^{-1}[(n,x)]p.$$

Then, $w_5 \nVdash_l \mathsf{K}_n[(n,a)]^{-1}[(n,c)]p$. Thus, by item (4) of Definition 3, there is a state w_5' of the left game such that $w_5 \sim_k w_5'$ and $w_5' \nVdash_l [(n,a)]^{-1}[(n,c)]p$. Statement $w_5 \sim_k w_5'$ implies that $w_5' = w_5$, see Figure 2. Hence, $w_5 \nVdash_l [(n,a)]^{-1}[(n,c)]p$. Then, by item (6) of Definition 3, there is a state w_5'' in the left game such that $(w_5'', \{(n,a)\}, w_5) \in M_l$ and $w_5'' \nVdash_l [(n,c)]p$. Statement $(w_5'', \{(n,a)\}, w_5) \in M_l$ implies that $w_5'' = w_1$, see Figure 2. Thus, $w_1 \nVdash_l [(n,c)]p$. Hence, by item (5) of Definition 3, there is a state w_1'' of the left game such that $(w_1, \{(n,c)\}, w_1'') \in M_l$ and $w_1'' \nVdash_l p$. Assumption $(w_1, \{(n,c)\}, w_1'') \in M_l$ implies that $w_1'' = w_3$, see Figure 2. Therefore, $w_3 \nVdash_l p$, which contradicts the definition of the left game, see Figure 2.

Case II: $w_2 \nVdash [(n,a)] \bigvee_{x\in\{b,c\}} \mathsf{K}_n[(n,a)]^{-1}[(n,x)]p$. The proof of this case is similar to the one above, except that it uses action b instead of action c and state w_6 instead of state w_5. □

Lemma 6. *$h(w_1) \nVdash_r \zeta$.*

Proof. Note that $h(w_1) = u_1$, see Figure 2. Suppose that

$$u_1 \Vdash \mathsf{K}_n[(n,a)] \bigvee_{x\in\{b,c\}} \mathsf{K}_n[(n,a)]^{-1}[(n,x)]p.$$

Thus,

$$u_1 \Vdash [(n,a)] \bigvee_{x\in\{b,c\}} \mathsf{K}_n[(n,a)]^{-1}[(n,x)]p$$

by item (4) of Definition 3. Note that $(u_1, \{(n,a)\}, u_5) \in M_r$, see Figure 2. Hence,

$$u_5 \Vdash \bigvee_{x\in\{b,c\}} \mathsf{K}_n[(n,a)]^{-1}[(n,x)]p$$

by item (5) of Definition 3. Then, either $u_5 \Vdash \mathsf{K}_n[(n,a)]^{-1}[(n,b)]p$ or $u_5 \Vdash \mathsf{K}_n[(n,a)]^{-1}[(n,c)]p$. We consider these two cases separately.

Case I: $u_5 \Vdash \mathsf{K}_n[(n,a)]^{-1}[(n,b)]p$. Thus, $u_5 \Vdash [(n,a)]^{-1}[(n,b)]p$ by item (4) of Definition 3. Note that $(u_1, \{(n,a)\}, u_5) \in M_r$, see Figure 2. Hence, $u_1 \Vdash [(n,b)]p$ by item (6) of Definition 3. Finally,

observe that $(u_1, \{(n, b)\}, u_4) \in M_r$. Then, $u_4 \Vdash p$ by item (5) of Definition 3, which contradicts the definition of the right game, see Figure 2.
Case II: $u_5 \Vdash \mathsf{K}_n[(n, a)]^{-1}[(n, c)]p$. Thus, $u_5 \Vdash [(n, a)]^{-1}[(n, c)]p$ by item (4) of Definition 3. Note that $(u_2, \{(n, a)\}, u_5) \in M_r$, see Figure 2. Hence, $u_2 \Vdash [(n, c)]p$ by item (6) of Definition 3. Finally, observe that $(u_2, \{(n, c)\}, u_4) \in M_r$. Then, $u_4 \Vdash p$ by item (5) of Definition 3, which again contradicts to the definition of the right game, see Figure 2. □

The next theorem follows from Lemma 4, Lemma 5, and Lemma 6.

Theorem 1. *Formula ζ is not definable in language Φ^-.* □

5 AXIOMS

In this section we propose a sound and complete logical system that describes the interplay between modalities K_C, $[s]$, and $[s]^{-1}$. In addition to the propositional tautologies in language Φ, our system consists of the following axioms:

(1) Truth: $\mathsf{K}_C\varphi \to \varphi$,
(2) Negative Introspection: $\neg\mathsf{K}_C\varphi \to \mathsf{K}_C\neg\mathsf{K}_C\varphi$,
(3) Distributivity: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$, where $\Box \in \{\mathsf{K}_C, [s], [s]^{-1}\}$,
(4) Monotonicity: $\mathsf{K}_C\varphi \to \mathsf{K}_D\varphi$, $[s]\varphi \to [t]\varphi$, and $[s]^{-1}\varphi \to [t]^{-1}\varphi$, where $C \subseteq D$ and $s \subseteq t$,
(5) Empty Set: $\mathsf{K}_\varnothing\varphi \to \Box\varphi$, where $\Box \in \{[s], [s]^{-1}\}$,
(6) Future-Past: $\neg\varphi \to [s]\neg[s]^{-1}\varphi$,
(7) Past-Future: $\neg\varphi \to [s]^{-1}\neg[s]\varphi$.

The Truth, the Negative Introspection, the Distributivity, and the Monotonicity axioms for modality K_C are the axioms of the epistemic logic S5 for distributed knowledge.

The Distributivity axiom for modality $[s]$ says that if strategy s guarantees $\varphi \to \psi$ and also guarantees φ, then it guarantees ψ. This axiom is true for the modality $[s]$ with an *explicit* strategy s. However, is not true in Marc Pauly's endogenous logic of coalition power, where it is replaced with the Cooperation axiom: $\mathsf{S}_C(\varphi \to \psi) \to (\mathsf{S}_D\varphi \to \mathsf{S}_{C\cup D}\psi)$, for disjoint sets C and D only. The Distributivity axiom for modality $[s]^{-1}$ says that if statement $\varphi \to \psi$ and statement φ have been true prior to any possible execution of strategy s, then statement ψ also has been true before an execution of strategy s.

The Monotonicity axiom for modality $[s]$ states that if strategy s guarantees φ, then so does any extension of this strategy to a larger coalition. The same axiom for modality $[s]^{-1}$ says that if statement φ has been true prior to any possible execution of strategy s (no matter what actions are taken by the agents not mentioned in s), then φ has also been true prior to any execution of strategy s accompanied by some additional actions.

The Empty Set axiom says that if condition φ is satisfied in each state of the game, then condition φ was satisfied in the past and is guaranteed to be satisfied in the future.

The Future-Past axiom states that if statement φ is not true now, then no matter what actions s are taken, statement $[s]^{-1}\varphi$ will not be true in the next state. Informally, it says that in the future one will not be able to change the present.

Finally, the Past-Future axiom says that if φ is not true after actions s are taken, then taking actions s could not have guaranteed φ. Informally, this axioms captures the famous saying "insanity is doing the same thing over and over again and expecting different results", which is often misattributed to Albert Einstein [15].

We write $\vdash \varphi$ if formula φ is provable in our system using the Modus Ponens and the Necessitation inference rules:

$$\frac{\varphi, \quad \varphi \to \psi}{\psi} \qquad\qquad \frac{\varphi}{\mathsf{K}_C\varphi}.$$

We write $X \vdash \varphi$ if formula φ is provable from the theorems of our logical system and an additional set of axioms X using only the Modus Ponens inference rule. The set X is consistent if there is no formula $\varphi \in \Phi$ such that $X \vdash \varphi$ and $X \vdash \neg\varphi$. We conclude this section with several lemmas about our logical system that will be used later in the proof of the completeness.

Lemma 7. *Inference rules* $\dfrac{\varphi}{[s]\varphi}$ *and* $\dfrac{\varphi}{[s]^{-1}\varphi}$ *are derivable.*

Proof. Formulae $[s]\varphi$ and $[s]^{-1}\varphi$ are derivable from formula φ through a combination of the Necessitation inference rule, the Empty Set axiom, and the Modus Ponens rule. □

In the next two lemmas we give examples of formal derivations in our logical system. In these examples, we assume that the Boolean constant $\bot$ is defined in our language in the standard way. Recall from the Section 3 that some combinations of actions might not lead to a next state. Similarly, some of the states in our games might have no previous states. Informally, the first lemma roughly states "if there is no past in the future, then there is no tomorrow".

Lemma 8. $\vdash [s][s]^{-1}\bot \to [s]\bot$.

Proof. We start by proving the first statement. Formula $\neg[s]^{-1}\bot \to ([s]^{-1}\bot \to \bot)$ is a propositional tautology. Thus, $\vdash [s](\neg[s]^{-1}\bot \to ([s]^{-1}\bot \to \bot))$ by Lemma 7. Hence, by the Distributivity axiom and the Modus Ponens inference rule, $\vdash [s]\neg[s]^{-1}\bot \to [s]([s]^{-1}\bot \to \bot)$. At the same time, formula $\neg\bot \to [s]\neg[s]^{-1}\bot$ is an instance of the Future-Past axiom. The last two formulae by the propositional reasoning imply that $\vdash [s]([s]^{-1}\bot \to \bot)$. Therefore, by the Distributivity axiom and the Modus Ponens rule, $\vdash [s][s]^{-1}\bot \to [s]\bot$. □

Informally, the second lemma states "if there was no future yesterday, then there was no yesterday". Its proof is dual to the first one in the sense that it replaces modality $[s]$ with $[s]^{-1}$, modality $[s]^{-1}$ with $[s]$, and the Future-Past axiom with the Past-Future axiom.

Lemma 9. $\vdash [s]^{-1}[s]\bot \to [s]^{-1}\bot$. □

The next two lemmas are well-known in modal logic. We reproduce their proofs here to keep the article self-contained.

Lemma 10. *If* $\varphi_1, \dots, \varphi_n \vdash \psi$ *and* $\Box$ *is one of the modalities* K_C, $[s]$, *and* $[s]^{-1}$, *then* $\Box\varphi_1, \dots, \Box\varphi_n \vdash \Box\psi$.

Proof. The deduction lemma for propositional logic applied n times to assumption $\varphi_1, \dots, \varphi_n \vdash \psi$ implies that $\vdash \varphi_1 \to (\cdots \to (\varphi_n \to \psi)\dots)$. Thus, $\vdash \Box(\varphi_1 \to (\cdots \to (\varphi_n \to \psi)\dots))$, by the Necessitation inference rule (if $\Box = \mathsf{K}_C$) or by Lemma 7 (if $\Box = [s]$ or $\Box = [s]^{-1}$). Hence, by the Distributivity axiom and the Modus Ponens inference rule,

$$\vdash \Box\varphi_1 \to \Box(\varphi_2 \cdots \to (\varphi_n \to \psi)\dots).$$

Then, $\Box\varphi_1 \vdash \Box(\varphi_2 \cdots \to (\varphi_n \to \psi)\dots)$ by the Modus Ponens inference rule. Thus, again by the Distributivity axiom and the Modus Ponens inference rule, $\Box\varphi_1 \vdash \Box\varphi_2 \to \Box(\varphi_3 \cdots \to (\varphi_n \to \psi)\dots)$. Therefore, $\Box\varphi_1, \dots, \Box\varphi_n \vdash \Box\psi$, by repeating the last two steps $n-2$ times. □

The next lemma states a well-known principle in the epistemic logic.

Lemma 11 (positive introspection). $\vdash \mathsf{K}_C\varphi \to \mathsf{K}_C\mathsf{K}_C\varphi$. □

6 SOUNDNESS

Soundness of the Truth, the Negative Introspection, the Distributivity, and the Monotonicity (for distributed knowledge modality K_C) axioms is well-known [19]. The soundness of the Distributivity and the Monotonicity axioms for modalities $[s]$ and $[s]^{-1}$ immediately follows from Definition 3. Below we prove the soundness of the remaining three axioms for an arbitrary state $w \in W$ of an arbitrary game $(W, \{\sim_a\}_{a\in\mathcal{A}}, \Delta, M, \pi)$.

Lemma 12. *If $w \Vdash \mathsf{K}_\varnothing\varphi$, then $w \Vdash [s]\varphi$ and $w \Vdash [s]^{-1}\varphi$.*

Proof. By Definition 3, assumption $w \Vdash \mathsf{K}_\varnothing\varphi$ implies that $u \Vdash \varphi$ for each state $u \in W$ of the game. Therefore, $w \Vdash [s]\varphi$ and $w \Vdash [s]^{-1}\varphi$ by Definition 3. □

Lemma 13. *If $w \nVdash \varphi$, then $w \Vdash [s]\neg[s]^{-1}\varphi$.*

Proof. Consider any complete action profile $\delta \in \Delta^{\mathcal{A}}$ and any state $u \in W$ such that $s \subseteq \delta$ and $(w, \delta, u) \in M$. By Definition 3, it suffices to show that $u \nVdash [s]^{-1}\varphi$, which is true again by Definition 3 and the assumption $w \nVdash \varphi$. □

Lemma 14. *If $w \nVdash \varphi$, then $w \Vdash [s]^{-1}\neg[s]\varphi$.*

Proof. Suppose that $w \nVdash [s]^{-1}\neg[s]\varphi$. Thus, by Definition 3, there is a state $u \in W$ and a complete action profile $\delta \in \Delta^{\mathcal{A}}$ such that $s \subseteq \delta$, $(u, \delta, w) \in M$, and $u \Vdash [s]\varphi$. Therefore, $w \Vdash \varphi$ by Definition 3. □

7 COMPLETENESS

The standard proof of the completeness for the epistemic logic of individual knowledge defines states as maximal consistent sets of formulae. It then proceeds to define $w_1 \sim_a w_2$ if sets w_1 and w_2 contain the same K_a-formulae. This construction does not work for the distributed knowledge modality because if two sets contain the same K_a- and K_b-formulae, then they do not necessarily contain the same $\mathsf{K}_{\{a,b\}}$-formulae. This issue can be solved using the tree construction below. The same construction has been previously used, for example, in [33].

For any maximal consistent set X_0, we define the canonical game $G(X_0) = (W, \{\sim_a\}_{a\in\mathcal{A}}, \Delta, M, \pi)$.

Definition 4. *Set of states W consists of all sequences $X_0, C_1, X_1, \dots, C_n, X_n$ where $n \geq 0$ such that*

(1) *X_i is a maximal consistent subset of Φ, for each $i \geq 1$,*
(2) *$C_i \subseteq \mathcal{A}$ is a coalition, for each $i \geq 1$,*
(3) *$\{\varphi \in \Phi \mid \mathsf{K}_{C_i}\varphi \in X_{i-1}\} \subseteq X_i$, for each $i \geq 1$.*

If x is a sequence $x_1, \dots, x_n$ and y is an arbitrary element, then by $x :: y$ and $hd(x)$ we mean sequence $x_1, \dots, x_n, y$ and element x_n respectively. For any states $w, u \in W$, if $u = w :: C :: X$ for some coalition C and some set X, then we say that states w and u are *adjacent*. The adjacency relation forms a tree structure on set W. We refer to the undirected edge connecting outcomes w and u as an edge *labeled* with each agent from set C, see Figure 3.

Definition 5. *For all states $w_1, w_2 \in W$ and any agent $a \in \mathcal{A}$, let $w_1 \sim_a w_2$ if each edge along the unique path connecting nodes w_1 and w_2 is labeled with agent a.*

Lemma 15. *$\mathsf{K}_C\varphi \in hd(w_1)$ iff $\mathsf{K}_C\varphi \in hd(w_2)$ for each formula $\varphi \in \Phi$, all coalitions C, and all outcomes $w_1, w_2 \in W$ such that $w_1 \sim_C w_2$.*

Proof. Assumption $w_1 \sim_C w_2$ by Definition 5, implies that each edge along the unique path between nodes w_1 and w_2 is labeled with all agents in coalition C. Thus, it suffices to show that

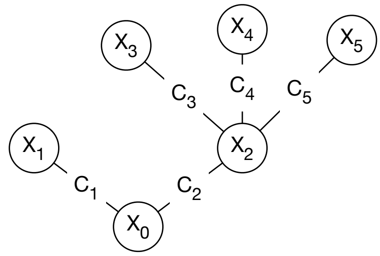

Fig. 3. A Fragment of the Tree of States.

$\mathsf{K}_C\varphi \in hd(w_1)$ iff $\mathsf{K}_C\varphi \in hd(w_2)$ for any two *adjacent* nodes along this path. Without loss of generality, let

$$\begin{aligned} w_1 &= X_0, C_1, X_1, \dots, C_{n-1}, X_{n-1} \\ w_2 &= X_0, C_1, X_1, \dots, C_{n-1}, X_{n-1}, C_n, X_n. \end{aligned}$$

The assumption that the edge between w_1 and w_2 is labeled with all agents in coalition C implies that $C \subseteq C_n$. We show next that $\mathsf{K}_C\varphi \in hd(w_1)$ iff $\mathsf{K}_C\varphi \in hd(w_2)$.
$(\Rightarrow)$: Suppose that $\mathsf{K}_C\varphi \in hd(w_1) = X_{n-1}$. Thus, $X_{n-1} \vdash \mathsf{K}_C\mathsf{K}_C\varphi$ by Lemma 11 and the Modus Ponens inference rule. Hence, $X_{n-1} \vdash \mathsf{K}_{C_n}\mathsf{K}_C\varphi$ by the Monotonicity axiom and the Modus Ponens inference rule because $C \subseteq C_n$. Then, $\mathsf{K}_{C_n}\mathsf{K}_C\varphi \in X_{n-1}$ because set X_{n-1} is maximal. Therefore, $\mathsf{K}_C\varphi \in X_n = hd(w_2)$ by Definition 4.
$(\Leftarrow)$: Suppose that $\mathsf{K}_C\varphi \notin hd(w_1) = X_{n-1}$. Thus, $\neg\mathsf{K}_C\varphi \in X_{n-1}$ because set X_{n-1} is maximal. Hence, $X_{n-1} \vdash \mathsf{K}_C\neg\mathsf{K}_C\varphi$ by the Negative Introspection axiom and the Modus Ponens inference rule. Then, $X_{n-1} \vdash \mathsf{K}_{C_n}\neg\mathsf{K}_C\varphi$ by the Monotonicity axiom and the Modus Ponens inference rule because $C \subseteq C_n$. Thus, $\mathsf{K}_{C_n}\neg\mathsf{K}_C\varphi \in X_{n-1}$ because set X_{n-1} is maximal. Hence, $\neg\mathsf{K}_C\varphi \in X_n$ by Definition 4. Therefore, $\mathsf{K}_C\varphi \notin X_n = hd(w_2)$ because X_n is consistent. □

We define the domain of actions Δ to be the extension of the set of actions Δ_0 by a single new action d_0. Note that action d_0 is not a part of the syntax of our logical system. Thus, no maximal consistent set contains a formula that uses action d_0. Informally, action d_0 means "abstain".

Definition 6. *Let* $\Delta = \Delta_0 \cup \{d_0\}$ *for some* $d_0 \notin \Delta_0$.

The definition of the canonical mechanism M closely resembles item 6 of Definition 3. It is interesting to note that although the mechanism is also referred to in item 5 of the same definition, modality $[s]^{-1}$ is not used to specify mechanism M.

Definition 7. *Mechanism* M *is the set of triples* (w, δ, u) *in* $W \times \Delta^{\mathcal{A}} \times W$ *such that for each formula* $[s]\varphi \in hd(w)$ *if* $s \subseteq \delta$, *then* $\varphi \in hd(u)$.

Definition 8. $\pi(p) = \{w \in W \mid p \in hd(w)\}$.

This concludes the definition of the canonical game $G(X_0)$. The next key step in the proof of the completeness is a "truth" lemma, which in our case is Lemma 20. Before proving this lemma we need to prove several auxiliary lemmas that will be used in the proof of Lemma 20.

Lemma 16. *For any formula* $\mathsf{K}_C\varphi \notin hd(w)$ *there is a state* $u \in W$ *such that* $w \sim_C u$ *and* $\varphi \notin hd(u)$.

Proof. Let X be the set $\{\neg\varphi\} \cup \{\psi \mid \mathsf{K}_C\psi \in hd(w)\}$.

Claim 1. *Set* X *is consistent.*

Proof of Claim. Suppose the opposite. Thus, there are such formulae

$$\mathsf{K}_C\psi_1, \dots, \mathsf{K}_C\psi_n \in hd(w), \tag{4}$$

that $\psi_1, \dots, \psi_m \vdash \varphi$. Hence, $\mathsf{K}_C\psi_1, \dots, \mathsf{K}_C\psi_m \vdash \mathsf{K}_C\varphi$ by Lemma 10. Then, $hd(w) \vdash \mathsf{K}_C\varphi$ because of statement (4). Therefore, $\mathsf{K}_C\varphi \in hd(w)$ due to the maximality of set $hd(w)$, which contradicts the assumption of the lemma. □

To finish the proof of the lemma, let X' be any maximal consistent extension of set X and u be the sequence $w :: C :: X'$. Note that $u \in W$ by Definition 4 and the choice of set X, set X', and sequence u. Finally, $\neg\varphi \in X \subseteq X' = hd(u)$ also by the choice of X, X', and u. Therefore, $\varphi \notin hd(u)$ because set $hd(u)$ is consistent. □

Lemma 17. *For any state $w \in W$ and any formula $[s]\varphi \notin hd(w)$ there is a complete action profile $\delta \in \Delta^{\mathcal{A}}$ and a state $u \in W$ such that $s \subseteq \delta$, $(w, \delta, u) \in M$, and $\varphi \notin hd(u)$.*

Proof. Let X be the set of formulae

$$\{\neg\varphi\} \cup \{\psi \mid [t]\psi \in hd(w), t \subseteq s\} \cup \{\chi \mid \mathsf{K}_\varnothing\chi \in hd(w)\}.$$

Claim 2. *Set X is consistent.*

Proof of Claim. Suppose the opposite. Hence, there are such formulae

$$[t_1]\psi_1, \dots, [t_m]\psi_m, \mathsf{K}_\varnothing\chi_1, \dots, \mathsf{K}_\varnothing\chi_n \in hd(w), \tag{5}$$

that $t_1, \dots, t_m \subseteq s$ and $\psi_1, \dots, \psi_m, \chi_1, \dots, \chi_n \vdash \varphi$. Thus, by Lemma 10,

$$[s]\psi_1, \dots, [s]\psi_m, [s]\chi_1, \dots, [s]\chi_n \vdash [s]\varphi.$$

Then, by the Monotonicity axiom and the Modus Ponens inference rule,

$$[t_1]\psi_1, \dots, [t_m]\psi_m, [s]\chi_1, \dots, [s]\chi_n \vdash [s]\varphi.$$

Hence, by the Empty Set axiom,

$$[t_1]\psi_1, \dots, [t_m]\psi_m, \mathsf{K}_\varnothing\chi_1, \dots, \mathsf{K}_\varnothing\chi_n \vdash [s]\varphi.$$

Thus, $hd(w) \vdash [s]\varphi$ by statement (5). Therefore, $[s]\varphi \in hd(w)$ due to the maximality of set $hd(w)$, which contradicts the assumption of the lemma. □

Let X' be any maximal consistent extension of set X and u be the sequence $w :: \varnothing :: X'$. Note that $u \in W$ by Definition 4, and the choice of set X, set X', and sequence u.

Also, let function δ be the functional relation $s \cup \{(a, d_0) \mid a \in \mathcal{A} \setminus Dom(s)\}$, where $Dom(s)$ is the domain of the functional relation s.

Claim 3. $(w, \delta, u) \in M$.

Proof of Claim. Consider any formula $[t]\psi \in hd(w)$ such that $t \subseteq \delta$. By Definition 7, it suffices to show that $\psi \in hd(u)$.

First, we show that $t \subseteq s$. Indeed, suppose there is a pair $(a, d) \in t$ such that $(a, d) \notin s$. Assumption $(a, d) \in t$ implies that $d \in \Delta_0$ by Definition 1 because $[t]\psi \in \Phi$. The same assumption $(a, d) \in t$ also implies that $(a, d) \in \delta$ because $t \subseteq \delta$. Statements $(a, d) \in \delta$ and $(a, d) \notin s$ imply that $d = d_0$ by the definition of profile δ. Thus, $d_0 = d \in \Delta_0$, which contradicts Definition 6. Therefore, $t \subseteq s$.

Statement $t \subseteq s$ and assumption $[t]\psi \in hd(w)$ imply $\psi \in X \subseteq X' = hd(u)$ by the choice of set X, set X', and sequence u. □

To finish the proof of the lemma, note that $\neg\varphi \in X \subseteq X' = hd(u)$ by the choice of X, X', and u. Therefore, $\varphi \notin hd(u)$ by the consistency of set $hd(u)$. □

Lemma 18. *For any $(w, \delta, u) \in M$ and any formula $[s]^{-1}\varphi \in hd(u)$, if $s \subseteq \delta$, then $\varphi \in hd(w)$.*

Proof. Suppose $\varphi \notin hd(w)$. Thus, $\neg\varphi \in hd(w)$ by the maximality of set $hd(w)$. Hence, by the Future-Past axiom and the Modus Ponens inference rule, $hd(w) \vdash [s]\neg[s]^{-1}\varphi$. Then, $[s]\neg[s]^{-1}\varphi \in hd(w)$ by the maximality of set $hd(w)$. Thus, $\neg[s]^{-1}\varphi \in hd(u)$ by Definition 7 and the assumption that $s \subseteq \delta$ and the assumption $(w, \delta, u) \in M$ of the lemma. Therefore, $[s]^{-1}\varphi \notin hd(u)$ because set $hd(u)$ is consistent. □

Lemma 19. *For any state $u \in W$ and any formula $[s]^{-1}\varphi \notin hd(u)$, there is a state $w \in W$ and a profile $\delta \in \Delta^{\mathcal{A}}$ such that $s \subseteq \delta$, $(w, \delta, u) \in M$, and $\varphi \notin hd(w)$.*

Proof. Let X be the set of formulae

$$\{\neg\varphi\} \cup \{\psi \mid [t]^{-1}\psi \in hd(u), t \subseteq s\} \cup \{\chi \mid \mathsf{K}_\varnothing\chi \in hd(u)\}.$$

Claim 4. *Set X is consistent.*

Proof of Claim. Suppose the opposite. Thus, there are such formulae

$$[t_1]^{-1}\psi_1, \dots, [t_m]^{-1}\psi_m, \mathsf{K}_\varnothing\chi_1, \dots, \mathsf{K}_\varnothing\chi_n \in hd(u), \tag{6}$$

that $t_1, \dots, t_m \subseteq s$ and $\psi_1, \dots, \psi_m, \chi_1, \dots, \chi_n \vdash \varphi$. Then, by Lemma 10,

$$[s]^{-1}\psi_1, \dots, [s]^{-1}\psi_m, [s]^{-1}\chi_1, \dots, [s]^{-1}\chi_n \vdash [s]^{-1}\varphi.$$

Hence, by the assumption $t_1, \dots, t_m \subseteq s$ and the Monotonicity axiom,

$$[t_1]^{-1}\psi_1, \dots, [t_m]^{-1}\psi_m, [s]^{-1}\chi_1, \dots, [s]^{-1}\chi_n \vdash [s]^{-1}\varphi.$$

Thus, by the Empty Set axiom,

$$[t_1]^{-1}\psi_1, \dots, [t_m]^{-1}\psi_m, \mathsf{K}_\varnothing\chi_1, \dots, \mathsf{K}_\varnothing\chi_n \vdash [s]^{-1}\varphi.$$

Then, $hd(u) \vdash [s]^{-1}\varphi$ because of statement (6). Hence, $[s]^{-1}\varphi \in hd(u)$ due to the maximality of set $hd(u)$, which contradicts the assumption of the lemma. Therefore, set X is consistent. □

Let X' be any maximal consistent extension of set X and w be the sequence $u :: \varnothing :: X'$. Note that $w \in W$ by Definition 4, and the choice of set X, set X', and sequence w. Also, let us define function δ to be the functional relation $s \cup \{(a, d_0) \mid a \in \mathcal{A} \setminus Dom(s)\}$, where $Dom(s)$ is the domain of the functional relation s.

Claim 5. *If $[t]\sigma \in hd(w)$ and $t \subseteq s$, then $\sigma \in hd(u)$.*

Proof of Claim. Suppose that $\sigma \notin hd(u)$. Thus, $\neg\sigma \in hd(u)$ because set $hd(u)$ is maximal. Hence, $hd(u) \vdash [t]^{-1}\neg[t]\sigma$ by the Past-Future axiom and the Modus Ponens rule. Then, $[t]^{-1}\neg[t]\sigma \in hd(u)$ because set $hd(u)$ is maximal. Thus, $\neg[t]\sigma \in X \subseteq X' = hd(w)$ by the choice of set X, set X', and sequence w. Therefore, $[t]\sigma \notin hd(w)$ because set $hd(w)$ is consistent, which contradicts the assumption of the claim. □

Claim 6. $(w, \delta, u) \in M$.

Proof of Claim. Consider a formula $[t]\sigma \in hd(w)$ such that $t \subseteq \delta$. By Definition 7, it suffices to show that $\sigma \in hd(u)$. We consider the following two cases separately.
Case I: $t \subseteq s$. Thus, the assumption $[t]\sigma \in hd(w)$ implies that $\sigma \in hd(u)$ by Claim 5.
Case II: There is a pair $(a, d) \in t \setminus s$. Thus, $d \in \Delta_0$ by Definition 7 and the assumption $[t]\sigma \in hd(w)$. At the same time, $t \subseteq \delta$ by the choice of formula $[t]\sigma$ and $\delta = s \cup \{(a, d_0) \mid a \in \mathcal{A} \setminus Dom(s)\}$ by the choice of δ. Hence, $d = d_0$ because $(a, d) \notin s$. Recall that $d \in \Delta_0$. Therefore, $d_0 \in \Delta_0$ which contradicts Definition 6. □

To finish the proof of the lemma, note that $\neg\varphi \in X \subseteq X' = hd(w)$ by the choice of set X, set X', and sequence w. Therefore, $\varphi \notin hd(w)$ due to the consistency of set $hd(w)$. □

LEMMA 20. *$\varphi \in hd(w)$ iff $w \Vdash \varphi$, for any state $w \in W$ and any formula $\varphi \in \Phi$.*

PROOF. We prove the statement of the lemma by structural induction on formula φ. If formula φ is a propositional variable, then the required follows from Definition 8 and Definition 3. The cases when formula φ is a negation or an implication follow from Definition 3 and the maximality and the consistency of formula φ in the standard way.

Suppose that formula φ has the form $\mathsf{K}_C\psi$.
($\Rightarrow$) Assume $\mathsf{K}_C\psi \in hd(w)$ and consider any state $u \in W$ such that $w \sim_C u$. By Definition 3, it suffices to show that $\psi \in hd(u)$. Indeed, by Lemma 15, assumption $\mathsf{K}_C\psi \in hd(w)$ implies that $\mathsf{K}_C\psi \in hd(u)$. Thus, $hd(u) \vdash \psi$ by the Truth axiom. Hence, $\psi \in hd(u)$ because set $hd(u)$ is maximal. Therefore, $h \Vdash \psi$ by the induction hypothesis.
($\Leftarrow$) Assume $\mathsf{K}_C\psi \notin hd(w)$. Then, by Lemma 16, there is a state $u \in W$ such that $w \sim_C u$ and $\psi \notin hd(u)$. Hence, $u \nVdash \psi$ by the induction hypothesis. Therefore, $w \nVdash \mathsf{K}_C\psi$ by Definition 3.

Suppose that formula φ has the form $[s]\psi$.
($\Rightarrow$) Assume $[s]\psi \in hd(w)$ and consider any profile $\delta \in \Delta^{\mathcal{A}}$ and any state $u \in W$ such that $s \subseteq \delta$ and $(w, \delta, u) \in M$. By Definition 3, it suffices to show that $\psi \in hd(u)$, which is true by Definition 7.
($\Leftarrow$) Assume $[s]\psi \notin hd(w)$. Thus, by Lemma 17, there is a profile $\delta \in \Delta^{\mathcal{A}}$ and a state $u \in W$ such that $s \subseteq \delta$, $(w, \delta, u) \in M$, and $\psi \notin hd(u)$. Hence, $u \nVdash \psi$ by the induction hypothesis. Therefore, $w \nVdash [s]\psi$ by Definition 3.

Finally, suppose that formula φ has the form $[s]^{-1}\psi$.
($\Rightarrow$) Assume $[s]^{-1}\psi \in hd(w)$ and consider any state $u \in W$ and any profile $\delta \in \Delta^{\mathcal{A}}$ such that $s \subseteq \delta$ and $(u, \delta, w) \in M$. By Definition 3, it suffices to show that $u \Vdash \psi$, which is true by Lemma 18 and the induction hypothesis.
($\Leftarrow$) Assume $[s]^{-1}\psi \notin hd(w)$. Thus, by Lemma 19, there is a state $u \in W$ and a complete action profile $\delta \in \Delta^{\mathcal{A}}$ such that $s \subseteq \delta$, $(u, \delta, w) \in M$, and $\psi \notin hd(u)$. Hence, $u \nVdash \psi$ by the induction hypothesis. Therefore, $w \nVdash [s]^{-1}\psi$ by Definition 3. □

THEOREM 2 (STRONG COMPLETENESS). *If $X \nvdash \varphi$, then there is a game $(W, \{\sim_a\}_{a\in\mathcal{A}}, \Delta, M, \pi)$ and a state $w \in W$ such that $w \Vdash \chi$ for all formulae $\chi \in X$ and $w \nVdash \varphi$.*

PROOF. Suppose $X \nvdash \varphi$. Consider any maximal consistent set X_0 such that $\{\neg\varphi\} \cup X \subseteq X_0$. By Definition 4, the single element sequence X_0 is a state of the canonical game $G(X_0)$. Also, $\varphi \notin X_0$ because $\neg\varphi \in X \subseteq X_0$ and set X_0 is consistent. Thus, $X_0 \nVdash \varphi$ by Lemma 20. Furthermore, $X_0 \nVdash \chi$ for each $\chi \in X$ also by Lemma 20 and because $X \subseteq X_0$. □

8 STRONG INCOMPLETENESS

Our results in the previous sections are based on the assumption of item (3) of Definition 2 that the set of actions Δ_0 of the language of our logical system does not have to include all actions from the set of actions Δ of a given game. In this section we consider the setting in which condition $\Delta_0 \subseteq \Delta$ in item (3) of Definition 2 is replaced with condition $\Delta_0 = \Delta$. In Theorem 3 below, we show that in such a setting, not only is our system not strongly complete, but neither is any other strongly sound logical system. We start with the definitions of strongly sound and strongly complete logical systems.

DEFINITION 9. *A logical system $\mathcal{L}$ is strongly sound when for any set of formulae $X \subseteq \Phi$, any formulae $\varphi \in \Phi$ such that $X \vdash_{\mathcal{L}} \varphi$, and any state $w \in W$ of a game $(W, \{\sim_a\}_{a\in\mathcal{A}}, \Delta_0, M, \pi)$, if $w \Vdash \chi$ for each $\chi \in X$, then $w \Vdash \varphi$.*

DEFINITION 10. *A logical system is strongly complete when $X \vdash_{\mathcal{L}} \varphi$ for any set of formulae $X \subseteq \Phi$ and any formula $\varphi \in \Phi$ such that for each state $w \in W$ of each game $(W, \{\sim_a\}_{a\in\mathcal{A}}, \Delta_0, M, \pi)$, if $w \Vdash \chi$ for each $\chi \in X$, then $w \Vdash \varphi$.*

Theorem 3. *If set Δ_0 is infinite, set of agents $\mathcal{A}$ is not empty, and the language Φ includes at least one propositional variable, then any strongly sound logical system is not strongly complete.*

Proof. Let a_0 be any agent from nonempty set $\mathcal{A}$ and p be any propositional variable. Suppose that there is a strongly sound and strongly complete logical system $\mathcal{L}$. Consider the set of formulae

$$X = \{[(a_0, d)]p \mid d \in \Delta_0\}. \tag{7}$$

Claim 7. $X \nvdash_{\mathcal{L}} [\varnothing]p$.

Proof of Claim. Suppose that $X \vdash_{\mathcal{L}} [\varnothing]p$. Thus, there is a finite list of actions $d_1, \dots, d_n \in \Delta_0$ such that

$$[(a_0, d_1)]p, \dots, [(a_0, d_n)]p \vdash_{\mathcal{L}} [\varnothing]p. \tag{8}$$

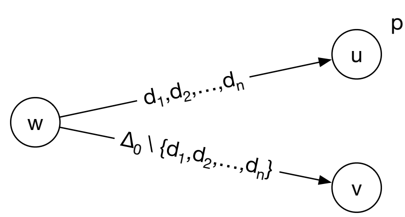

Fig. 4. A Game.

Consider game $(W, \{\sim_a\}_{a\in\mathcal{A}}, \Delta_0, M, \pi)$ depicted in Figure 4. This game has three states: w, u, and v. Only actions of agent a_0 affect the transitions of the game from one state to another. Namely, if in state w agent a_0 uses one of the actions $d_1, \dots, d_n$, then the game transitions into state u. If in state w agent a_0 uses any other action, then the game transitions into state v. The game never transitions from state u and v. Propositional variable p is only satisfied in state u.

Note that $w \Vdash [(a_0, d_i)]p$ for each $i \le n$ by item 5 of Definition 3. Thus,

$$w \Vdash [\varnothing]p \tag{9}$$

by Definition 10, strong soundness of system $\mathcal{L}$, and statement (8). Recall that set Δ_0 is infinite by the assumption of the theorem. Let $d' \in \Delta_0 \setminus \{d_1, \dots, d_n\}$. Consider any action profile δ such that $\delta(a_0) = d'$. Then, $(w, \delta, v) \in M$, see Figure 4. Therefore, $v \Vdash p$ by statement (9) and item (5) of Definition 3, which contradicts to the definition of the game, see Figure 4. □

By Definition 10, Claim 7 and the assumption that logical system $\mathcal{L}$ is strongly complete imply that there is a state $w \in W$ of a game $(W, \{\sim_a\}_{a\in\mathcal{A}}, \Delta_0, M, \pi)$ such that,

$$w \Vdash \chi, \qquad \text{for each formula } \chi \in X \tag{10}$$

and $w \nVdash [\varnothing]p$. By item (5) of Definition 3, the latter statement implies that there is a state $u \in W$ and a profile $\delta \in \Delta_0^{\mathcal{A}}$ such that $(w, \delta, u) \in M$ and $u \nVdash p$. Note that $[(a, \delta(a))]p \in X$ by equation (7). Also, $w \nVdash [(a, \delta(a))]p$ by the same item (5) of Definition 3. Therefore, there is a formula $\chi \in X$ such that $w \nVdash \chi$, which contradicts to statement (10). □

Finally, note that if set Δ_0 is finite and we consider only games such that $\Delta = \Delta_0$, then our logical system is still not complete. For example, if $\Delta_0 = \{d_1, d_2\}$ and $a \in \mathcal{A}$, then formula $[(a, d_1)]\varphi \to ([(a, d_2)]\varphi \to [\varnothing]\varphi)$ is true in all states of each game in which $\Delta = \Delta_0$. It is easy to show that this formula is not true in games that have at least one more action in addition to d_1 and d_2. Therefore, the formula is not provable in our logical system because our system is sound.

9 CONCLUSION

Most logical systems for reasoning about actions, strategies, and coalition power focus on possible future outcomes or “look ahead”. In this article we developed a symmetric system that also allows to make statements about the past based on the current state and the past actions or to “look back”. Such an extension, in combination with the knowledge modality, allows to capture many new properties of actions. Among such properties the most interesting is the ability of coalitions to learn from past experiences. Our main technical contribution is a sound and complete logical system capturing the interplay between “look ahead”, “look back”, and distributed knowledge modalities. We also show that the ability to learn from past experiences cannot be expressed without “look back” modality. Finally, we considered the class of games that can only have actions that are explicitly mentioned in the language and proved that no strongly sound logical system can be strongly complete with respect to this class.

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