Budget-Constrained Coalition Strategies with Discounting

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Abstract

Discounting future costs and rewards is a common practice in accounting, game theory, and machine learning. In spite of this, existing logics for reasoning about strategies with cost and resource constraints do not account for discounting. The article proposes a sound and complete logical system for reasoning about budget-constrained strategic abilities that incorporates discounting into its semantics.

1 Introduction

Several logical systems for reasoning about agent and coalition power in game-like settings have been previously proposed. Among them are coalition logics [19, 20], ATL [7], ATEL [21], ATLES [22], know-how logics [1, 23, 15, 13, 17, 16, 18], and STIT [8, 24, 14]. Some of these systems have been extended to incorporate resources and costs of actions [11, 6, 10, 3, 2, 12, 4, 5]. Even in the case of multi-step actions, these systems treat current and future costs equally.

At the same time, in game theory, accounting, and machine learning, costs of multi-step transitions are often discounted to reflect the fact that future costs and earnings have lesser present values. Thus, there is a gap between the way resources and costs currently are treated in logic and the way they are accounted for in other fields. To address this gap, in this article we propose a sound and complete logic of coalition power whose semantics incorporates discounting. Although we formulate our work in terms of cost, it could be applied to any other resource measured in non-negative real numbers. It can also be straightforwardly extended to vectors of non-negative real numbers in order to incorporate multiple resources.

As an example, consider a single-player game depicted in Figure 1. This game has four game states w, u, v, and s and a single terminal state t. Propositional variable p is true in game states w, u, and v and is false in game state s. We assume that the values of propositional variables are not defined in the terminal state t. The agent a has multiple actions in each game state. These actions are depicted in Figure 1 using directed edges. The cost of each action to agent a is shown as a label on the directed edge. For instance, the directed edge from state w to state u with label 2 means that the agent a has an action with cost 2 to transition the game from state w to state u. Transitioning to the terminal state t represents the termination of the game.

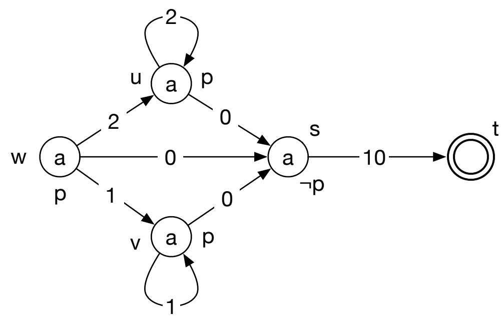

Figure 1: A game.

Note that in state w the agent has two strategies to maintain condition p indefinitely. The first strategy consists of transitioning the game to state u at cost 2 and then repeatedly applying the action with cost 2 to keep the game in state u. Without discounting, the cost of this strategy is $2 + 2 + 2 + \cdots = +\infty$. The agent also has another strategy to maintain condition p that consists of transitioning to state v at cost 1 and then keeping the game in state v with recurrent cost 1. Intuitively, the second strategy is less expensive than the first because each step costs half as much. However, formally, the cost of the second strategy without discounting is the same as the first one: $1 + 1 + 1 + \cdots = +\infty$.

The problem that we observe here is not specific to costs of strategies. A similar situation also appears in repetitive games, accounting, and reinforcement machine learning algorithms based on Markov decision processes. One of the solutions commonly used to resolve this problem is discounting[1]. It consists of counting the cost on the first step at the nominal value, the cost on the second step with a discount factor $\gamma \in (0, 1)$, the cost on the third step with discount factor γ^2, etc. With discounting, the total cost of our first strategy is

$$2 + 2\gamma + 2\gamma^2 + 2\gamma^3 + \cdots = \frac{2}{1-\gamma},$$

while the cost of our second strategy is

$$1 + 1\gamma + 1\gamma^2 + 1\gamma^3 + \cdots = \frac{1}{1-\gamma}.$$

Since $\frac{1}{1-\gamma} < \frac{2}{1-\gamma}$, we can say that with discounting the second strategy is less expensive than the first. In the rest of this article, we assume a fixed discount factor $\gamma \in (0, 1)$.

[1]The other solution is using the limit of mean cost. Discounting is used when the agents care more about short-term costs than long-term ones, whereas the limit of mean cost is used to treat short-term and long-term costs equally.

2 Outline

The rest of this article is structured as follows: In the next section, we introduce a class of games that is used later to define the semantics of our logical system. Section 4 defines the language of our system. Section 5 gives the discounting-based semantics of this language. Section 6 shows that the properties of strategies with discounting depend on whether we consider strategies with or without perfect recall. Section 7 lists and discusses the axioms of our logical system for the strategies with perfect recall. Section 9 proves the completeness of our system. Sections 10 through 12 discuss various possible extensions of our logical system. Section 13 concludes. A preliminary version of this work, with a partial proof of the completeness and without discussion sections 10 through 12, appeared as [9].

3 Game Definition

Throughout the article, we assume a fixed nonempty set of propositional variables and a fixed set of agents $\mathcal{A}$. By a coalition we mean any subset of $\mathcal{A}$. By $X^{\mathcal{A}}$ we mean the set of all functions from set $\mathcal{A}$ to a set X.

The class of games that we consider is specified below.

Definition 1 *A game is a tuple $(W, t, \Delta, \varepsilon, M, \pi)$, where*

1. *W is a set of **game states**,*
2. *$t \notin W$ is a **terminal** state, by W^t we denote the set of all states $W \cup \{t\}$,*
3. *Δ is an arbitrary set called **domain of actions**,*
4. *$\varepsilon \in \Delta$ is a **zero-cost action**,*
5. *$M \subseteq W \times \Delta^{\mathcal{A}} \times [0, \infty)^{\mathcal{A}} \times W^t$ is a relation called **mechanism**, such that*
   (a) *for each tuple $(w, \delta, u, w') \in M$ and each agent $a \in \mathcal{A}$, if $\delta(a) = \varepsilon$, then $u(a) = 0$,*
   (b) *for each state $w \in W$ and each **complete action profile** $\delta \in \Delta^{\mathcal{A}}$, there is a function $u \in [0, +\infty)^{\mathcal{A}}$ and a state $w' \in W^t$ such that $(w, \delta, u, w') \in M$,*
6. *π is a **valuation function** that maps propositional variables into subsets of W.*

Intuitively, the mechanism is a set of all quadruples (w, δ, u, v) such that the game might transition from state w to state v under action profile δ at costs to the individual agents specified by function u. The word "complete" in the above definition refers to the fact that function δ assigns actions to *all* agents.

The defined above games are similar to resource-bounded action frames, which are the semantics of Resource-Bounded Coalition Logic (RBCL) [6]. In particular, both of them have a zero-cost action.

However, there are several differences between these two classes of models. Unlike RBCL frames, our games have only one resource that we call “cost”. We do this for the sake of presentation simplicity. Multiple resources could be incorporated into our system without any significant changes to the results in this article. RBCL allows only non-negative *integer* resource requirements, while costs in our games are non-negative *real* numbers. RBCL assigns a unique cost to each *action*, while our games assign cost to each *transition*. As a result, the cost to the agent in our setting depends not only on the action of the agent but also on the actions of the other agents. This is similar to how the utility function of an agent in game theory is a function of the complete action profile, not just of the action of that agent. We achieve this by including the cost of the transition for each agent as the third component of a tuple from the mechanism relation.

Furthermore, the RBCL frames are deterministic while our games are not deterministic because we represent the mechanism as a relation, not a function. Unlike RBCL frames, our games can be terminated by the agents. In order to make our semantics more general, the games are terminated through a transition to a terminal state. Such transitions allow the agents to be charged upon the termination of a game. This ability is significant for our proof of completeness.

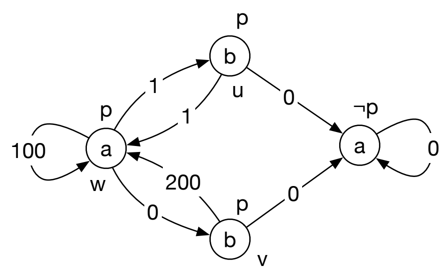

Figure 2: A game. The unreachable terminal state t is not shown in the diagram.

The game depicted in Figure 1 has only one player. Figure 2 depicts a two-player game. Note that our general notion of the game captured in Definition 1 allows each agent to influence the outcome of each transition and imposes a cost for each transition on each agent. For the sake of simplicity, the game depicted in Figure 2 designates a single “dictator” agent in each state. For example, in state w, the dictator is agent a. The dictator is solely responsible for the choice of the next state and bears all the costs associated with the transition. In the diagram, the dictator is specified inside each state's circle. Note that in state w agent a has a strategy to maintain condition p at cost $100 + 100\gamma + 100\gamma^2 + \cdots = \frac{100}{1-\gamma}$ to herself and cost $0 + 0\gamma + 0\gamma^2 + \cdots = 0$ to agent b.

Definition 2 *A finite sequence* $w_0, \delta_0, u_0, w_1, \ldots, \delta_{n-1}, u_{n-1}, w_n$ *is called a play of a game* $(W, t, \Delta, \varepsilon, M, \pi)$*, if*

1. $w_i \in W$ *for* $0 \leq i < n$ *and* $w_n \in W^t$,

2. $\delta_i \in \Delta^{\mathcal{A}}$, *where* $0 \leq i < n$,

3. $u_i \in [0, \infty)^{\mathcal{A}}$ *is a* ***cost function****, where* $0 \leq i < n$,

4. $(w_{i-1}, \delta_{i-1}, u_{i-1}, w_i) \in M$, *where* $1 \leq i \leq n$.

The set of all plays of a given game is denoted by *Play*.

4 Syntax

The language Φ of our system is defined by the grammar

$$\varphi := p \mid \neg\varphi \mid \varphi \to \varphi \mid [C]_x\varphi,$$

where p is a propositional variable, C is a coalition, and x is a "constraint" function from set C to $[0, +\infty)$. We read $[C]_x\varphi$ as "coalition C has a strategy to maintain condition φ at individual cost no more than $x(a)$ to each member $a \in C$".

If C is a coalition $\{a_1, \dots, a_n\}$ and x is a function from set C to $[0, +\infty)$ such that $x(a_i) = x_i$ for each $i \leq n$, then we use shorthand notation $[a_1, \dots, a_n]_{x_1,\dots,x_n}\varphi$ to refer to formula $[C]_x\varphi$.

Definition 3 *For any real* $\mu > 0$ *and any formula* $\varphi \in \Phi$*, formula* φ/μ *is defined recursively as follows:*

1. $p/\mu \equiv p$, *for any propositional variable* p,

2. $(\neg\varphi)/\mu \equiv \neg(\varphi/\mu)$,

3. $(\varphi \to \psi)/\mu \equiv (\varphi/\mu) \to (\psi/\mu)$,

4. $([C]_x\varphi)/\mu \equiv [C]_{x/\mu}(\varphi/\mu)$.

For example, $([a, b]_{4,6}\neg[b, c]_{8,2}\, p)/2 = [a, b]_{2,3}\neg[b, c]_{4,1}\, p$.

5 Semantics

In this section we define the semantics of our logical system.

Definition 4 *An action profile of a coalition* C *is a function from set* C *to set* Δ.

Definition 5 *A strategy of a coalition* C *is a function from set* $C \times Play$ *to set* Δ.

Note that each strategy takes into account not just the current state but the whole play. Thus, the strategies that we consider are *perfect recall* strategies. We discuss this in detail in the next section.

Definition 6 *A play $w_0, \delta_0, u_0, w_1, \dots, u_{n-1}, w_n \in Play$ satisfies strategy s of a coalition C if for each i such that $0 \le i < n$ and each agent $a \in C$,*

$$\delta_i(a) = s(a, (w_0, \delta_0, u_0, w_1, \dots, u_{i-1}, w_i)).$$

For any functions x and y, we write $x \le_C y$ if $x(a) \le y(a)$ for each $a \in C$. We define notation $x =_C y$ similarly.

Definition 7 *For each formula $\varphi \in \Phi$ and each state $w \in W$ of a game $(W, t, \Delta, \varepsilon, M, \pi)$, satisfaction relation $w \Vdash \varphi$ is defined recursively as follows:*

1. *$w \Vdash p$, if $w \in \pi(p)$,*
2. *$w \Vdash \neg\varphi$, if $w \nVdash \varphi$,*
3. *$w \Vdash \varphi \to \psi$, if $w \nVdash \varphi$ or $w \Vdash \psi$,*
4. *$w \Vdash [C]_x\varphi$ if there is a strategy s of coalition C such that for any play $w_0, \delta_0, u_0, w_1, \dots, u_{n-1}, w_n \in Play$ that satisfies strategy s, if $w = w_0$, then*
   (a) *$\sum_{i=0}^{n-1} u_i\gamma^i \le_C x$ and*
   (b) *if $w_n \ne t$, then $w_n \Vdash \varphi/\gamma^n$.*

To understand why item 4(b) of the above definition uses formula φ/γ^n instead of formula φ, let us consider an example of a formula $\varphi \equiv [D]_y\psi$. Note that formula $[C]_x[D]_y\psi$ states that coalition C can maintain at cost x the ability of coalition D to maintain ψ at cost y. Consider the hypothetical case where C, at cost x to C, will be maintaining this ability of D for, say, 10 transitions. The formula $[C]_x[D]_y\psi$ states that after 10 moves coalition D should be able to take over and maintain condition ψ at cost y to D. Given that in our setting the costs are discounted, an important question is whether y is measured in *today's* money or *future* money. Note that y in future money is $y\gamma^{10}$ in today's money. On the other hand, y in today's money is y/γ^{10} in future money. In this article we decided to measure all costs in today's money. Thus, cost y in $[C]_x[D]_y\psi$ refers to costs in today's money (in state w_0 of Definition 7). In future money (in state w_n), the same cost is y/γ^n. As a result, item 4(b) of Definition 7 uses formula φ/γ^n instead of just φ. We further discuss future money in Section 12.

Consider again the game depicted in Figure 2. As discussed earlier, in state w, single-agent coalition $\{a\}$ has a strategy to maintain condition p by looping in state w at recurrent cost 100. The total cost of this strategy is $100 + 100\gamma + 100\gamma^2 + \cdots = \frac{100}{1-\gamma}$. Thus, $w \Vdash [a]_{100/(1-\gamma)}\, p$. In the same game, single-agent coalition $\{b\}$ also has a strategy to maintain condition p. The strategy consists in pushing the game back to state w each time when agent a transitions the game out of state w either into state u or state v. The cost of the "pushing back" action from state u and v is 1 and 200, respectively. Hence, the total cost to agent b could be no more than $0+200\gamma+0+200\gamma^3+0+\cdots = 200\gamma/(1-\gamma^2)$. Then, $w \Vdash [b]_{200\gamma/(1-\gamma^2)}\, p$. Finally, note that if agents a and b decide to cooperate, then maintaining condition p becomes significantly less expensive for both of them because they can alternate the state of the game between states w and u. The total cost of the joint strategy to agent a is $1+0+\gamma^2+0+\gamma^3+\cdots = \frac{1}{1-\gamma^2}$ and to agent b is $0+\gamma+0+\gamma^3+\cdots = \frac{\gamma}{1-\gamma^2}$. Therefore, $w \Vdash [a,b]_{1/(1-\gamma^2),\gamma/(1-\gamma^2)}\, p$.

6 Perfect Recall Assumption

Definition 5 specifies a strategy of a coalition as a function that assigns an action to each member of a coalition based on a play of the game. In other words, any strategy has access to the whole history of the game rather than just to the current state. Such strategies are often referred to as *perfect recall strategies*. As the next example shows, perfect recall strategies might have different discounted costs than memoryless strategies for the same condition to maintain in the same game.

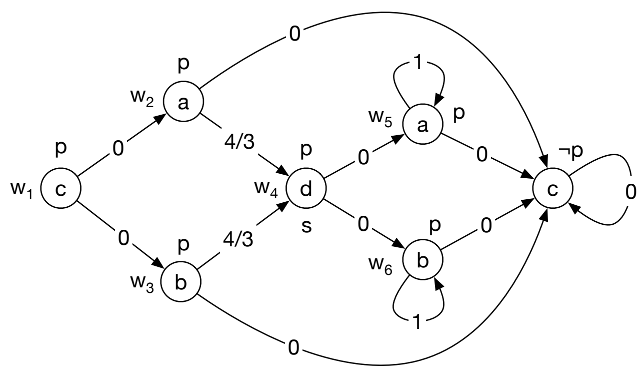

Figure 3: A game. The unreachable terminal state t is not shown in the diagram.

Consider the game depicted in Figure 3 and assume, for this example only, that $\gamma = 2/3$. Suppose that coalition $\{a, b, d\}$ wants to maintain condition p starting from state w_1.

Since agent c is not a member of the coalition, the coalition has no control whether the system transitions from state w_1 to state w_2 or w_3. Once the system is either in state w_2 or state w_3, in order to maintain the condition p, agent a or agent b, respectively, will have to transition the game to state w_4 at cost $\frac{4}{3}\gamma = \frac{4}{3} \cdot \frac{2}{3} = \frac{8}{9}$ to the agent. In state w_4, the coalition faces a choice between (i) transitioning game into state w_5 in which agent a encounters cost

$$1\gamma^3 + 1\gamma^4 + \cdots = \frac{\gamma^3}{1-\gamma} = \frac{(2/3)^3}{1/3} = \frac{8}{9}$$

to maintain p and (ii) transitioning game into state w_6 in which agent b encounters the same cost $1\gamma^3 + 1\gamma^4 + \cdots = \frac{8}{9}$ to maintain condition p.

If agent d has a perfect recall, then she can balance the costs between agents a and b by transitioning to state w_6 if the game transitioned to w_4 from state w_2 and transitioning to state w_5 if the game transitioned to w_4 from state w_3. This way, agents a and b encounter the same total costs 8/9:

$$w_1 \Vdash [a, b, d]_{8/9,8/9,0}\, p.$$

At the same time, if agent d does not have memory about the previous state of the game, then either agent a or b might encounter a total cost as high as $8/9 + 8/9 = 16/9$ while executing the coalition strategy to maintain condition p:

$$w_1 \Vdash [a, b, d]_{16/9,16/9,0}\, p.$$

In this article, we consider discounted costs under perfect recall assumption for all agents.

7 Axioms

In this section, we introduce a logical system describing the properties of coalition power modality $[C]_x\varphi$. In addition to propositional tautologies in language Φ, the system contains the following axioms:

1. Reflexivity: $[C]_x\varphi \to \varphi$,
2. Cooperation: $[C]_x(\varphi \to \psi) \to ([D]_y\varphi \to [C \cup D]_{x\cup y}\psi)$, where $C \cap D = \varnothing$,
3. Monotonicity: $[C]_x\varphi \to [C]_y\varphi$, where $x \leq_C y$,
4. Transitivity: $[C]_x\varphi \to [C]_x[C]_x\varphi$.

Recall that the value of discount factor γ has been fixed at the end of Section 1. It is worth noting that this factor does not appear explicitly in any of the above axioms.

The Reflexivity axiom says that if coalition C can maintain condition φ at discounted cost x starting from the current state, then condition φ must be true in the current state. The Cooperation axiom states that if coalitions C and D are disjoint, coalition C can maintain condition $\varphi \to \psi$ at cost x, and D can maintain condition φ at cost y, then together they can maintain condition ψ at cost $x \cup y$. Here, by $x \cup y$ we mean the union of two functions with disjoint domains. The Monotonicity axiom states that if a coalition can maintain a condition at some cost, then it can maintain the same condition at any larger cost.

The assumption of the Transitivity axiom states that coalition C has a strategy, say s, to maintain condition φ at cost x in perpetuity. The conclusion states that the same coalition can, at cost x, maintain its own ability to maintain φ at cost x. To achieve this, coalition C can use the same strategy s. Indeed, assume that coalition C used strategy s for some number of steps at cost $x' \leq_C x$ in today's money. Thus, it should be able to keep using it at cost at most $x - x' \leq_C x$

in today's money to maintain φ. Note that it is crucial for this argument that all costs are computed in today's money. Furthermore, as we observe in Section 12, the Transitivity axiom is not sound if the cost in the internal modality is measured in future money.

We write $\vdash \varphi$ and say that formula φ is a *theorem* if φ is derivable from the above axioms using the Modus Ponens and the Necessitation inference rules:

$$\frac{\varphi, \varphi \to \psi}{\psi} \qquad\qquad \frac{\varphi}{[C]_x\varphi}.$$

In addition to unary relation $\vdash \varphi$, we also consider binary relation $X \vdash \varphi$. Let $X \vdash \varphi$ if formula φ is provable from *the theorems* of our logical system and the set of additional assumptions X using *only* the Modus Ponens inference rule.

Lemma 1 *If $\varphi_1, \dots, \varphi_n \vdash \psi$ and sets $C_1,\dots,C_n$ are pairwise disjoint, then*

$$[C_1]_{x_1}\varphi_1, \dots, [C_n]_{x_n}\varphi_n \vdash [C_1 \cup \dots \cup C_n]_{x_1\cup\dots\cup x_n}\psi.$$

PROOF. Apply the deduction lemma n times to the assumption $\varphi_1, \dots, \varphi_n \vdash \psi$. Then, $\vdash \varphi_1 \to (\dots \to (\varphi_n \to \psi))$. Thus,

$$\vdash [\varnothing]_0(\varphi_1 \to (\dots \to (\varphi_n \to \psi)))$$

by the Necessitation inference rule. Hence,

$$\vdash [C_1]_{x_1}\varphi_1 \to [C_1]_{x_1}(\varphi_2 \dots \to (\varphi_n \to \psi))$$

by the Cooperation axiom and the Modus Ponens inference rule. Then,

$$[C_1]_{x_1}\varphi_1 \vdash [C_1]_{x_1}(\varphi_2 \dots \to (\varphi_n \to \psi))$$

by the Modus Ponens inference rule. Thus, again by the Cooperation axiom and the Modus Ponens inference rule,

$$[C_1]_{x_1}\varphi_1 \vdash [C_2]_{x_2}\varphi_2 \to [C_1 \cup C_2]_{x_1\cup x_2}(\varphi_3 \dots \to (\varphi_n \to \psi)).$$

Therefore, $[C_1]_{x_1}\varphi_1, \dots, [C_n]_{x_n}\varphi_n \vdash [C_1 \cup \dots \cup C_n]_{x_1\cup\dots\cup x_n}\psi$ by repeating the last two steps $n-2$ times. ⊠

Lemma 2 *If $\varphi_1/\gamma, \dots, \varphi_n/\gamma \vdash \psi/\gamma$, then $\varphi_1, \dots, \varphi_n \vdash \psi$.*

PROOF. Note that if a sequence of formulae $\chi_1, \dots, \chi_n$ is a derivation in our logical system, then for each real number $\mu > 0$, sequence $\chi_1/\mu, \dots, \chi_n/\mu$ is also a derivation. Hence, for any formulae $\varphi_1, \dots, \varphi_n, \psi \in \Phi$, if $\varphi_1, \dots, \varphi_n \vdash \psi$, then $\varphi_1/\mu, \dots, \varphi_n/\mu \vdash \psi/\mu$. Let $\mu = \gamma^{-1}$. Thus, for any formulae $\varphi_1, \dots, \varphi_n, \psi \in \Phi$, if $\varphi_1/\gamma, \dots, \varphi_n/\gamma \vdash \psi/\gamma$, then $\varphi_1, \dots, \varphi_n \vdash \psi$. ⊠

Lemma 3 $\vdash [C]_x\varphi \to [D]_y\varphi$*, where $C \subseteq D$ and $x \leq_C y$.*

PROOF. Let y' be the restriction of function y to set $D \setminus C$. Then, by the Cooperation axiom and the assumption $C \subseteq D$,

$$\vdash [D \setminus C]_{y'}(\varphi \to \varphi) \to ([C]_x\varphi \to [D]_{y' \cup x}\varphi).$$

Note that $\varphi \to \varphi$ is a propositional tautology. Hence, by the Necessitation inference rule $\vdash [D \setminus C]_{y'}(\varphi \to \varphi)$. Then, $\vdash [C]_x\varphi \to [D]_{y' \cup x}\varphi$ by the Modus Ponens inference rule. Note that $\vdash [D]_{y' \cup x}\varphi \to [D]_y\varphi$ by the Monotonicity axiom and the assumption $x \leq_C y$ of the lemma. Therefore, $\vdash [C]_x\varphi \to [D]_y\varphi$ by propositional reasoning. ⊠

8 Soundness

In this section we prove soundness of each of our axioms as a separate lemma. In these lemmas we assume that w is an arbitrary game state of a game $(W, t, \Delta, \varepsilon, M, \pi)$.

Lemma 4 *If $w \Vdash [C]_x\varphi$, then $w \Vdash \varphi$.*

PROOF. Single-element sequence w is a play by Definition 2. Thus, by item 4 of Definition 7, the assumption $w \Vdash [C]_x\varphi$ implies that $w \Vdash \varphi/\gamma^0$. Hence, $w \Vdash \varphi/1$. Therefore, $w \Vdash \varphi$ by Definition 3. ⊠

Lemma 5 *If $w \Vdash [C]_x\varphi$ and $x \leq_C y$, then $w \Vdash [C]_y\varphi$.*

PROOF. By item 4 of Definition 7, the assumption $w \Vdash [C]_x\varphi$ implies that there is a strategy s of coalition C such that for any play $w_0, \delta_0, u_0, w_1, \dots, u_{n-1}, w_n \in \mathit{Play}$ that satisfies strategy s, if $w = w_0$, then

1. $\sum_{i=0}^{n-1} u_i\gamma^i \leq_C x$ and
2. if $w_n \neq t$, then $w_n \Vdash \varphi/\gamma^n$.

Note that condition 1 above implies that $\sum_{i=0}^{n-1} u_i\gamma^i \leq_C y$ by the assumption $x \leq_C y$ of the lemma. Therefore, $w \Vdash [C]_y\varphi$ again by item 4 of Definition 7. ⊠

Lemma 6 *If $w \Vdash [C]_x(\varphi \to \psi)$, $w \Vdash [D]_y\varphi$, and coalitions C and D are disjoint, then $w \Vdash [C \cup D]_{x \cup y}\psi$.*

PROOF. By item 4 of Definition 7, the assumption $w \Vdash [C]_x(\varphi \to \psi)$ implies that there is a strategy s_1 of coalition C such that for any play

$$w_0, \delta_0, u_0, w_1, \dots, u_{n-1}, w_n \in \mathit{Play}$$

that satisfies strategy s_1, if $w = w_0$, then

1. $\sum_{i=0}^{n-1} u_i\gamma^i \leq_C x$ and
2. if $w_n \neq t$, then $w_n \Vdash (\varphi \to \psi)/\gamma^n$.

Similarly, the assumption $w \Vdash [D]_y\varphi$ implies that there is a strategy s_2 of coalition C such that for any play $w_0, \delta_0, u_0, w_1, \dots, u_{n-1}, w_n \in \mathit{Play}$ that satisfies strategy s_2, if $w = w_0$, then

3. $\sum_{i=0}^{n-1} u_i\gamma^i \leq_D y$ and
4. if $w_n \neq t$, then $w_n \Vdash \varphi/\gamma^n$.

Consider strategy s of coalition $C \cup D$ such that

$$s(a, \lambda) = \begin{cases} s_1(a, \lambda), & \text{if } a \in C, \\ s_2(a, \lambda), & \text{if } a \in D, \end{cases}$$

for any play $\lambda \in \mathit{Play}$. Note that strategy s is well-defined because sets C and D are disjoint by the assumption of the lemma.

Consider any play $w_0, \delta_0, u_0, w_1, \dots, u_{n-1}, w_n \in \mathit{Play}$ satisfying strategy s. Thus, by Definition 6, this play satisfies strategies s_1 and s_2. Note that conditions 1 and 3 above imply that $\sum_{i=0}^{n-1} u_i\gamma^i \leq_{C\cup D} x \cup y$. Suppose that $w_n \neq t$. By item 4 of Definition 7, it suffices to show that $w_n \Vdash \psi/\gamma^n$. Indeed, condition 2 above implies that $w_n \Vdash (\varphi \to \psi)/\gamma^n$. Thus, $w_n \Vdash \varphi/\gamma^n \to \psi/\gamma^n$ by Definition 3. Therefore, $w_n \Vdash \psi/\gamma^n$ by item 3 of Definition 7 and condition 4 above. ⊠

The next auxiliary lemma follows from Definition 3.

Lemma 7 *$\varphi/(\gamma\gamma') = (\varphi/\gamma)/\gamma'$.* ⊠

Lemma 8 *If $w \Vdash [C]_x\varphi$, then $w \Vdash [C]_x[C]_x\varphi$.*

PROOF. By item 4 of Definition 7, the $w \Vdash [C]_x\varphi$ implies that there is a strategy s of coalition C such that for any play $w_0, \delta_0, u_0, w_1, \dots, u_{n-1}, w_n \in \mathit{Play}$ that satisfies strategy s, if $w = w_0$, then

$$\sum_{i=0}^{n-1} u_i\gamma^i \leq_C x \tag{1}$$

and

$$\text{if } w_n \neq t, \text{ then } w_n \Vdash \varphi/\gamma^n. \tag{2}$$

Consider any play $w'_0, \delta'_0, u'_0, w'_1, \dots, u'_{m-1}, w'_m \in \mathit{Play}$ that satisfies strategy s such that $w = w'_0$. By the same item 4 of Definition 7, it suffices to show that

$$\sum_{i=0}^{m-1} u'_i\gamma^i \leq_C x \tag{3}$$

and

$$\text{if } w'_m \neq t, \text{ then } w'_m \Vdash ([C]_x\varphi)/\gamma^m. \tag{4}$$

Note that statement (3) follows from assumption (1). Thus, it is enough to prove statement (4). Suppose $w'_m \neq t$, then, by Definition 3, it suffices to show that

$$w'_m \Vdash [C]_{x/\gamma^m}(\varphi/\gamma^m). \tag{5}$$

Consider strategy

$$s'(a,(w_0,\delta_0,u_0,w_1,\dots,u_{n-1},w_n)) = \tag{6}$$
$$= \begin{cases} s(a,(w'_0,\delta'_0,w'_1,\dots,u'_{m-1}, \\ \qquad w'_m,\delta_0,u_0,\dots,u_{n-1},w_n)), & \text{if } w'_m = w_0 \\ \varepsilon, & \text{otherwise.} \end{cases}$$

Consider any play $w''_0,\delta''_0,u''_0,w''_1,\dots,u''_{k-1},w''_k \in Play$ that satisfies strategy s' such that $w'_m = w''_0$. By the same item 4 of Definition 7, to prove statement (5) it suffices to show that

$$\sum_{i=0}^{k-1} u''_i\gamma^i \leq_C x/\gamma^m$$

and if $w''_k \neq t$, then $w''_k \Vdash ([C]_{x/\gamma^m}(\varphi/\gamma^m))/\gamma^k$. Both of these facts follow from the three claims below.

Claim 1 *The play $w'_0,\delta'_0,u'_0,w'_1,\dots,u'_{m-1},w'_m,\delta''_0,u''_0,w''_1,\dots,\ u''_{k-1},w''_k$ satisfies strategy s.*

Proof of Claim. The statement of the claim follows from Definition 6, equation (6), and the assumption that the play $w''_0,\delta''_0,u''_0,w''_1,\dots,u''_{k-1},w''_k$ satisfies strategy s'. ⊠

Claim 2 $\sum_{i=0}^{k-1} u''_i\gamma^i \leq_C x/\gamma^m$.

Proof of Claim. By statement (1) and Claim 1,

$$\sum_{i=0}^{m-1} u'_i\gamma^i + \sum_{i=m}^{m+k-1} u''_{i-m}\gamma^i \leq_C x.$$

Functions $u'_0,\dots,u'_{m-1}$ are non-negative by item 5 of Definition 1. Hence, $\sum_{i=m}^{m+k-1} u''_{i-m}\gamma^i \leq_C x$. Thus, $\gamma^m\sum_{i=0}^{k} u''_i\gamma^i \leq_C x$. Then, $\sum_{i=0}^{k-1} u''_i\gamma^i \leq_C x/\gamma^m$. ⊠

Claim 3 *If $w''_k \neq t$, then $w''_k \Vdash ([C]_{x/\gamma^m}(\varphi/\gamma^m))/\gamma^k$.*

Proof of Claim. By Claim 1, the play $w'_0, \delta'_0, u'_0, w'_1, \dots, u'_{m-1}, w'_m, \delta''_0, u''_0, w''_1, \dots, u''_{k-1}, w''_k$ satisfies strategy s. Then, $w''_k \Vdash ([C]_x\varphi)/\gamma^{m+k}$ by equation (2) and the assumption $w''_k \neq t$. Hence $w''_k \Vdash (([C]_x\varphi)/\gamma^m)/\gamma^k$ by Lemma 7. Therefore, $w''_k \Vdash ([C]_{x/\gamma^m}(\varphi/\gamma^m))/\gamma^k$ by item 4 of Definition 3. ⊠

This concludes the proof of the lemma. ⊠

9 Completeness

In this section, we prove the completeness of our logical system. We start the proof by defining the canonical game $(W, t, \Delta, \varepsilon, M, \pi)$. The set W is the set of all maximal consistent sets of formulae in language Φ, and t is an arbitrary element such that $t \notin W$. Let ε be an arbitrary element such that $\varepsilon \notin \Phi$ and the set of actions Δ be $\Phi \cup \{\varepsilon\}$.

Definition 8 *Mechanism M is the set of all quadruples $(w, \delta, u, w') \in W \times \Delta^{\mathcal{A}} \times [0,\infty)^{\mathcal{A}} \times W^t$ such that for each formula $[C]_x\varphi \in w$, if $\delta(a) = [C]_x\varphi$ for each agent $a \in C$, then*

1. *$u \leq_C x$ and*
2. *if $w' \neq t$, then $([C]_{x-u}\varphi)/\gamma \in w'$.*

Informally, action $\delta(a) = [C]_x\varphi$ of an agent $a \in C$ means "as a part of coalition C, I request to maintain condition φ at individual cost $x(b)$ to each member $b \in C$". In order for the request to be valid, it should be submitted by all members of coalition C. Even if all members of coalition C submit the request, it is enforced by the mechanism only if formula $[C]_x\varphi$ belongs to the current state w. Condition 1 of Definition 8 stipulates that although the mechanism is free to set the cost u of the transition below what the members of the coalition offered to pay, the mechanism *cannot overcharge* them. If the mechanism decides to charge members of the coalition the amount u for transition to state w', then it also must *provide the opportunity* for the members to continue to maintain the condition φ at cost $x - u$. The latter is captured by condition 2 of Definition 8.

Definition 9 $\pi(p) = \{w \in W \mid p \in w\}$.

This concludes the definition of the canonical game $(W, t, \Delta, \varepsilon, M, \pi)$. As usual, the key step in proving the completeness theorem is an "induction" (or "truth") lemma, which in our case is Lemma 11. Lemma 9 and Lemma 10 below are two auxiliary lemmas that capture the two directions of the induction lemma in the case when formula φ has the form $[C]_x\psi$.

Lemma 9 *For each state $w \in W^t$ and each formula $[C]_x\varphi \in w$, there is strategy s of coalition C such that, for each play $w_0, \delta_0, u_0, w_1, \dots, u_{n-1}, w_n$ satisfying strategy s, if $w = w_0$, then*

1. *$\sum_{i=0}^{n-1} u_i\gamma^i \leq_C x$ and*

2. *if* $w_n \neq t$, *then* $\varphi/\gamma^n \in w_n$.

Proof. Let action $s(a,\lambda)$ for any agent $a \in C$ and any play

$$\lambda = w_0, \delta_0, u_0, w_1, \dots, u_{n-1}, w_n$$

be defined[2] as follows:

$$s(a,\lambda) = \begin{cases} ([C]_{x-z}\varphi)/\gamma^n, & \text{if } z \leq_C x, \\ \top, & \text{otherwise,} \end{cases} \tag{7}$$

where $z = \sum_{i=0}^{n-1} u_i \gamma^i$.

Consider an arbitrary play $w_0, \delta_0, u_0, w_1, \dots, u_{n-1}, w_n$ satisfying strategy s such that $w = w_0$. It will be sufficient to show that conditions 1 and 2 of the lemma hold for this play.

Claim 4 *For each* $a \in C$ *and each* k *such that* $0 \leq k \leq n$,

1. $\sum_{i=0}^{k-1} u_i \gamma^i \leq_C x$ *and*

2. *if* $w_k \neq t$, *then* $\left([C]_{x-\sum_{i=0}^{k-1} u_i\gamma^i}\varphi\right)/\gamma^k \in w_k$.

Proof of Claim. We prove the claim by induction on integer k. If $k = 0$, then $\sum_{i=0}^{k-1} u_i \gamma^i = 0 \leq_C x$ by the definition of language Φ because $[C]_x\varphi$ is a formula. Also,

$$\left([C]_{x-\sum_{i=0}^{k-1} u_i\gamma^i}\varphi\right)/\gamma^k = \left([C]_{x-0}\varphi\right)/\gamma^0 = [C]_x\varphi \in w_0$$

by the assumption $[C]_x\varphi \in w$ of the lemma and the assumption $w = w_0$.

Suppose $k > 0$. Then, $(w_{k-1}, \delta_{k-1}, u_{k-1}, w_k) \in M$ by Definition 2, the assumption of the lemma that $w_0, \delta_0, u_0, w_1, \dots, u_{n-1}, w_n$ is a play, and the assumption of the claim that $k \leq n$. Thus, $w_{k-1} \neq t$ by item 5 of Definition 1. Hence, by the induction hypothesis,

$$\sum_{i=0}^{k-2} u_i \gamma^i \leq_C x, \tag{8}$$

$$\left([C]_{x-\sum_{i=0}^{k-2} u_i\gamma^i}\varphi\right)/\gamma^{k-1} \in w_{k-1}. \tag{9}$$

By Definition 6 (step i), equation (8) and equation (7) (step ii), and item 4 of Definition 3 (step iii),

$$\begin{aligned} \delta_{k-1}(a) &\overset{\text{i}}{=} s(a, (w_0, \delta_0, u_0, w_1, \dots, u_{k-2}, w_{k-1})) \\ &\overset{\text{ii}}{=} ([C]_{x-\sum_{i=0}^{k-2} u_i\gamma^i}\varphi)/\gamma^{k-1} \\ &\overset{\text{iii}}{=} [C]_{(x-\sum_{i=0}^{k-2} u_i\gamma^i)/\gamma^{k-1}}(\varphi/\gamma^{k-1}). \end{aligned} \tag{10}$$

[2]Informally, strategy s always requests to maintain condition φ using remaining budget $x - z$.

At the same time, by item 4 of Definition 3 (step iv) and equation (9) (step v),

$$[C]_{(x-\sum_{i=0}^{k-2} u_i\gamma^i)/\gamma^{k-1}}(\varphi/\gamma^{k-1}) \stackrel{\text{iv}}{=} \left([C]_{x-\sum_{i=0}^{k-2} u_i\gamma^i}\varphi\right)/\gamma^{k-1} \stackrel{\text{v}}{\in} w_{k-1}. \tag{11}$$

Also, $(w_{k-1}, \delta_{k-1}, u_{k-1}, w_k) \in M$ by Definition 2 and the assumption that $w_0, \delta_0, u_0, w_1, \dots, u_{n-1}, w_n$ is a play. Thus, by Definition 8 and statements (11) and (10),

1. $u_{k-1} \leq_C \left(x - \sum_{i=0}^{k-2} u_i\gamma^i\right)/\gamma^{k-1}$ and
2. if $w_k \neq t$, then $([C]_{((x-\sum_{i=0}^{k-2} u_i\gamma^i)/\gamma^{k-1}-u_{k-1})}(\varphi/\gamma^{k-1}))/\gamma \in w_k$.

Thus, by the laws of algebra and item 4 of Definition 3,

1. $u_{k-1}\gamma^{k-1} \leq_C x - \sum_{i=0}^{k-2} u_i\gamma^i$ and
2. if $w_k \neq t$, then $([C]_{((x-\sum_{i=0}^{k-2} u_i\gamma^i)-u_{k-1}\gamma^{k-1})}\varphi)/\gamma^k \in w_k$.

The last two statements imply, respectively, parts 1 and 2 of the claim. ⊠

The statement of the lemma follows from the above claim when $k = n$. The first part follows immediately. To show the second part, note that by Definition 3, item 2 of the claim implies $\left([C]_{(x-\sum_{i=0}^{n-1} u_i\gamma^i)/\gamma^n}(\varphi/\gamma^n)\right) \in w_n$. Thus, $w_n \vdash \varphi/\gamma^n$ by the Reflexivity axiom and the Modus Ponens inference rule. Therefore, $\varphi/\gamma^n \in w_n$ because set w_n is maximal. ⊠

Lemma 10 *For each state $w \in W$, each formula $[C]_x\varphi \notin w$, and each action profile α of coalition C, there is a complete action profile δ, a cost function $u \in [0, +\infty)^{\mathcal{A}}$, and a state $w' \in W^t$ such that $\alpha =_C \delta$, $(w, \delta, u, w') \in M$, and either (i) $u \not\leq_C x$ or (ii) $w' \neq t$ and $\varphi/\gamma \notin w'$.*

Proof. Define the complete action profile

$$\delta(a) = \begin{cases} \alpha(a), & \text{if } a \in C, \\ \top, & \text{otherwise} \end{cases} \tag{12}$$

and cost function[3]

$$u(a) = \begin{cases} y(a), & \text{if } \alpha(a) = [D]_y\psi \text{ for some } [D]_y\psi \in \Phi, \\ & \text{where } a \in C \text{ and } y(a) > x(a), \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\alpha =_C \delta$. We consider the following two cases:

Case I: $u(a) = 0$ for each agent $a \in C$. Consider set

$$X = \{\neg(\varphi/\gamma)\} \cup \{([D]_y\psi)/\gamma \mid [D]_y\psi \in w, D \subseteq C, \forall a \in D(\alpha(a) = [D]_y\psi)\}.$$

[3]The choice of function u is perhaps the most unexpected step in our proof. Informally, if agent a is "bluffing" and is offering to pay more than $x(a)$, then function u charges the agent the amount she offered to pay, $y(a)$. If the agent makes a "modest" offer of no more than $x(a)$, then she is not charged at all.

Claim 5 *Set X is consistent.*

PROOF OF CLAIM. Suppose set X is not consistent. Thus, there are formulae

$$[D_1]_{y_1}\psi_1,\dots,[D_n]_{y_n}\psi_n \in w \tag{13}$$

such that

$$D_1,\dots,D_n \subseteq C, \tag{14}$$
$$\alpha(a) = [D_i]_{y_i}\psi_i \qquad \forall i \le n \;\; \forall a \in D_i, \tag{15}$$

and

$$([D_1]_{y_1}\psi_1)/\gamma,\dots,([D_n]_{y_n}\psi_n)/\gamma \vdash \varphi/\gamma. \tag{16}$$

Without loss of generality, we can assume that formulae

$$([D_1]_{y_1}\psi_1)/\gamma,\dots,([D_n]_{y_n}\psi_n)/\gamma$$

are distinct. Thus, formulae $[D_1]_{y_1}\psi_1,\dots,[D_n]_{y_n}\psi_n$ are also distinct by Definition 3. Hence, sets $D_1,\dots,D_n$ are pairwise disjoint due to assumption (15).

By Lemma 2, statement (16) implies that

$$[D_1]_{y_1}\psi_1,\dots,[D_n]_{y_n}\psi_n \vdash \varphi.$$

Then, by Lemma 1 and because sets $D_1,\dots,D_n$ are pairwise disjoint,

$$[D_1]_{y_1}[D_1]_{y_1}\psi_1,\dots,[D_n]_{y_n}[D_n]_{y_n}\psi_n \vdash [D_1\cup\dots\cup D_n]_{y_1\cup\dots\cup y_n}\varphi.$$

Thus, by the Transitivity axiom and the Modus Ponens inference rule applied n times,

$$[D_1]_{y_1}\psi_1,\dots,[D_n]_{y_n}\psi_n \vdash [D_1\cup\dots\cup D_n]_{y_1\cup\dots\cup y_n}\varphi.$$

Notice that $y_i(a) \le x(a)$ for any $i \le n$ and any agent $a \in D_i$. Indeed, suppose that $y_i(a) > x(a)$. Hence $u(a) = y_i(a)$ by the choice of cost function u and statements (14) and (15). Thus, $u(a) > x(a)$. Then, $u(a) > 0$ because function x is non-negative by the assumption $[C]_x\varphi \in \Phi$, which contradicts the assumption $u(a) = 0$ of the case. Hence, by Lemma 3 and the Modus Ponens inference rule,

$$[D_1]_{y_1}\psi_1,\dots,[D_n]_{y_n}\psi_n \vdash [C]_x\varphi.$$

Then, $w \vdash [C]_x\varphi$ by the assumption (13). Thus, $[C]_x\varphi \in w$ because set w is maximal, which contradicts the assumption $[C]_x\varphi \notin w$ of the lemma. ⊠

Let w' be any maximal consistent extension of set X. Note that $\neg(\varphi/\gamma) \in X \subseteq w'$ by the choice of sets X and w'. Thus, $\varphi/\gamma \notin w'$ because set w' is consistent.

Claim 6 $(w,\delta,u,w') \in M$.

Proof of Claim. Consider any formula $[D]_y\psi \in w$ such that

$$\delta(a) = [D]_y\psi \quad \text{for each agent} \quad a \in D. \tag{17}$$

By Definition 8, it suffices to show that $u \le_D y$ and $([D]_{y-u}\psi)/\gamma \in w'$. We consider the following two cases:
Case Ia: $D \subseteq C$. Thus, $\alpha(a) = \delta(a) = [D]_y\psi$ for each agent $a \in D$ by equation (12) and assumption (17). Hence, $([D]_y\psi)/\gamma \in X$ by the choice of set X. Then, $([D]_{y-u}\psi)/\gamma \in X$ by the assumption of Case I that $u =_C 0$ and the assumption $D \subseteq C$ of Case Ia. Therefore, $([D]_{y-u}\psi)/\gamma \in w'$ by the choice of set w'. Additionally, $u =_D 0 \le_D y$ because $0 \le_D y$ by the definition of Φ.
Case Ib: There is an agent $a \in D\setminus C$. Hence, $\top = \delta(a) = [D]_y\psi$ by equation (12) and assumption (17). Therefore, formula $[D]_y\psi$ is identical to formula $\top$, which is a contradiction. ⊠
Note that $\neg(\varphi/\gamma) \in X \subseteq w'$ by the choice of sets X and w'. Therefore, $\varphi/\gamma \notin w'$ because set w' is consistent.

Case II: $u(a) \neq 0$ for at least one agent $a \in C$. Thus, $u(a) = y(a) > x(a)$ by the choice of function u. Hence, $u \not\le_C x$. Choose w' to be the terminal state t.

Claim 7 $(w, \delta, u, w') \in M$.

Proof of Claim. Consider any formula $[D]_y\psi \in w$ such that $\delta(a) = [D]_y\psi$ for each agent $a \in D$. By Definition 8 and because $w' = t$, it suffices to show that $u \le_D y$. Recall that $0 \le_D y$ because $[D]_y\psi$ is a formula. Therefore, $u \le_D y$ by the choice of function u. ⊠
This concludes the proof of the lemma. ⊠

The next lemma is usually referred to as an "induction" or "truth" lemma.

Lemma 11 *$w \Vdash \varphi$ iff $\varphi \in w$ for each state $w \in W$ and each formula $\varphi \in \Phi$.*

Proof. We prove the statement by induction on the structural complexity of formula φ. If φ is a propositional variable, then the required follows from item 1 of Definition 7 and Definition 9. If formula φ is a negation or an implication, then the statement of the lemma follows from items 2 and 3 of Definition 7, the induction hypothesis, and the maximality and consistency of set w in the standard way.

Suppose that formula φ has the form $[C]_x\psi$.
$(\Rightarrow)$: Assume that $[C]_x\psi \notin w$. Consider any strategy s of coalition C. Define action profile α of coalition C such that, for each agent $a \in C$,

$$\alpha(a) = s(a, w_0). \tag{18}$$

By Lemma 10, there is a complete action profile δ, a cost function $u \in [0, +\infty)^{\mathcal{A}}$, and a state $w' \in W^t$ such that $\alpha =_C \delta$, $(w, \delta, u, w') \in M$, and

$$\text{either (i) } u \not\le_C x \text{ or (ii) } w' \neq t \text{ and } \psi/\gamma \notin w'. \tag{19}$$

Consider play w, δ, w'. This play satisfies strategy s by Definition 6, the assumption $\alpha =_C \delta$, and equation (18). Then, by item 4 of Definition 7, to prove $w \nVdash [C]_x\psi$, it suffices to show that either (i) $u \not\leq_C x$ or (ii) $w' \neq t$ and $w' \nVdash \psi/\gamma$. Note that this statement is true by statement (19) and the induction hypothesis.
$(\Leftarrow)$: Suppose that $[C]_x\varphi \in w$. By Lemma 9, there is a strategy s of coalition C such that, for each play $w_0, \delta_0, u_0, w_1, \dots, u_{n-1}, w_n$ satisfying strategy s, if $w = w_0$, then

1. $\sum_{i=0}^{n-1} u_i\gamma^i \leq_C x$ and
2. if $w_n \neq t$, then $\varphi/\gamma^n \in w_n$.

Therefore, $w \Vdash [C]_x\varphi$ by item 4 of Definition 7 and the induction hypothesis. ⊠

Theorem 1 *If $X \nvdash \varphi$, then there is a state w of a game such that $w \Vdash \chi$ for each $\chi \in X$ and $w \nVdash \varphi$.*

Proof. Suppose that $X \nvdash \varphi$. Let w be any maximal consistent extension of set $X \cup \{\neg\varphi\}$. Note that w is a state of the canonical game. Then, $w \Vdash \chi$ for each $\chi \in X$ and $w \Vdash \neg\varphi$ by Lemma 11. Therefore, $w \nVdash \varphi$ by Definition 7. ⊠

10 Negative costs

In item 5 of Definition 1, we assumed that cost function u has only non-negative values. This assumption is significant because the Transitivity axiom of our logical system does not hold for the games with negative costs. To observe this, consider the single-agent game depicted in Figure 4 and assume that discount

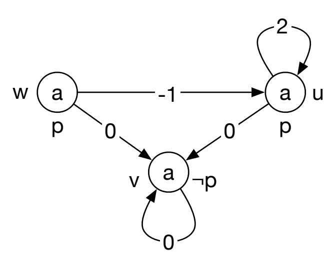

Figure 4: A game. The unreachable terminal state t is not shown in the diagram.

factor γ is equal to $\frac{1}{2}$. In state w of this game, agent a has a strategy to maintain condition p by first transitioning from state w to state u and then looping in state u indefinitely. The cost of this strategy is

$$-1 + 2\gamma + 2\gamma^2 + 2\gamma^3 + \cdots = -1 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots = 1.$$

Hence, $w \Vdash [a]_1 p$ by item 4 of Definition 7.

To finish the example, it suffices to show that $w \nVdash [a]_1[a]_1 p$. Towards a contradiction, suppose that $w \Vdash [a]_1[a]_1 p$. Thus, by item 4 of Definition 7, in state w agent a has a strategy to maintain condition $[a]_1 p$. Note that the strategy cannot transition the game from state w to state v because $v \nVdash p$, see Figure 4. Hence, it must transition the game into state u. Then, $u \Vdash ([a]_1 p)/\gamma$ by item 4 of Definition 7. Thus, $u \Vdash [a]_{1/\gamma} p$ by Definition 3. Hence, $u \Vdash [a]_2 p$ by the assumption $\gamma = \frac{1}{2}$. Then, by item 4 of Definition 7, in state u agent a has a strategy to maintain condition p at cost 2. Note that the only strategy in state u to maintain condition p is to loop in state u indefinitely. The cost of this strategy is

$$2 + 2\gamma + 2\gamma^2 + 2\gamma^3 + \cdots = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots = 4,$$

which is a contradiction because $4 > 2$.

Informally, this example exploits the fact that the strategy from state w is able to "earn" 1 during the transition from w to u and to use it to offset the cost of looping in state u. The strategy that originates in state u does not have access to this "earned" money.

11 Achieving in One Step

Language Φ of the proposed logical system could be extended by an additional modality $\langle\langle C \rangle\rangle_x \varphi$ that stands for "coalition C can achieve φ in one step using a strategy with costs at most x". Here, by a strategy of a coalition C we mean any function $s \in \Delta^C$ that assigns an action to each member of the coalition.

Just as in Definition 7, we assume that all values represent costs in today's money. As a result, the semantics of the modality $\langle\langle C \rangle\rangle_x \varphi$ in the definition below uses formula φ/γ instead of φ.

Definition 10 *$w \Vdash \langle\langle C \rangle\rangle_x \varphi$ if there is an action profile $s \in \Delta^C$ of coalition C such that for any $(w, \delta, u, v) \in M$, if $s = \delta_C$, then $u \leq_C x$ and $v \Vdash \varphi/\gamma$.*

A non-trivial property that captures the interplay between modalities $[C]_x \varphi$ and $\langle\langle C \rangle\rangle_x \varphi$ is

$$\langle\langle C \rangle\rangle_x [C]_y \varphi \to (\varphi \to [C]_{x+y\gamma} \varphi).$$

Note that unlike the axioms of our logical system listed in Section 7, this property explicitly incorporates the discount factor γ.

12 Future Money

As we discussed in Section 5, the semantics of our logical system assumes that all costs in a formula refer to today's money. One can also consider a future-money modality $[\![C]\!]_x \varphi$ whose semantics would be defined as follows:

Definition 11 *$w \Vdash [\![C]\!]_x\varphi$ if there is a strategy s of coalition C such that for any play $w_0, \delta_0, u_0, w_1, \dots, u_{n-1}, w_n \in Play$ that satisfies strategy s, if $w = w_0$, then*

1. *$\sum_{i=0}^{n-1} u_i\gamma^i \leq_C x$ and*
2. *if $w_n \neq t$, then $w_n \Vdash \varphi$.*

Note that the only difference between the definition above and item 4 of Definition 7 is the absence of division by γ^n in item 2.

It is interesting to observe that modality $[\![C]\!]_x\varphi$ does not satisfy the transitivity axiom either. To show this, let us consider a single-agent game depicted

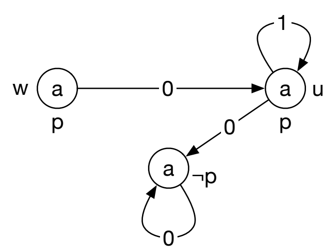

Figure 5: A game. The unreachable terminal state t is not shown in the diagram.

in Figure 5 and assume that discount factor γ is equal to $\frac{1}{2}$. Note that in state w agent a has a strategy to maintain condition p by first transitioning to state u and then looping in state u indefinitely. The total costs of this strategy is

$$0 + 1\gamma + 1\gamma^2 + 1\gamma^3 + \cdots = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1.$$

Thus, $w \Vdash [\![a]\!]_1\, p$. To finish the example, it suffices to show that $w \nVdash [\![a]\!]_1[\![a]\!]_1\, p$. Towards a contradiction, suppose that $w \Vdash [\![a]\!]_1[\![a]\!]_1\, p$. Thus, by Definition 11, in state w there is a strategy of agent a to maintain condition p at cost 1. Note that this strategy from state w can only transition the game into state u, see Figure 5. Thus, $u \Vdash [\![a]\!]_1\, p$ by Definition 11. Hence, again by Definition 11, in state u agent a has a strategy to maintain condition p at cost 1. Note that the only strategy in state u to maintain condition p is to loop in state u indefinitely. The cost of this strategy is

$$1 + 1\gamma + 1\gamma^2 + 1\gamma^3 + \cdots = 1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2,$$

which is a contradiction because $2 > 1$.

One might wonder if the Transitivity axiom for modality $[\![C]\!]_x\varphi$ holds in a modified form that takes into account the discount factor in one of the following forms:

$$w \Vdash [\![C]\!]_x\, \varphi \to [\![C]\!]_x [\![C]\!]_{x/\gamma}\, \varphi,$$
$$w \Vdash [\![C]\!]_x\, \varphi \to [\![C]\!]_x(([\![C]\!]_x \varphi)/\gamma).$$

As it turns out, neither of these principles is universally true because

$$w \nVdash [\![a]\!]_{1/2}\, p \to [\![a]\!]_{1/2} [\![a]\!]_1\, p$$

for the game depicted in Figure 6. The proof of this is very similar to the one above.

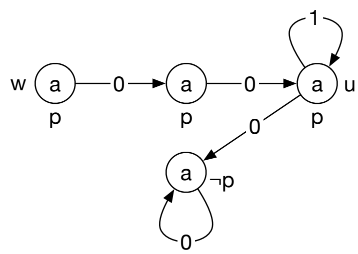

Figure 6: A game. The unreachable terminal state t is not shown in the diagram.

A transitivity-like property for modality $[\![C]\!]_x\varphi$ could be stated if the language is further extended by modality $[\![C]\!]^k_x\varphi$, which stands for "coalition C has a strategy to maintain condition φ for k steps at costs x". It can be formally defined as follows.

Definition 12 *$w \Vdash [\![C]\!]^k_x\varphi$ if there is a strategy s of coalition C such that for any play $w_0, \delta_0, u_0, w_1, \dots, u_{n-1}, w_n \in Play$ that satisfies strategy s, if $w = w_0$ and $n \leq k$, then*

1. *$\sum_{i=0}^{n-1} u_i \gamma^i \leq_C x$ and*
2. *if $w_n \neq t$, then $w_n \Vdash \varphi$.*

The transitivity-like property that uses modalities $[\![C]\!]_x\varphi$ and $[\![C]\!]^k_x\varphi$ is

$$[\![C]\!]_x\varphi \to [\![C]\!]^k_x(([\![C]\!]_x\varphi)/\gamma^k).$$

13 Conclusion

In this article we proposed a coalition power logic whose semantics incorporates discounting. Such a logical system could be potentially applied to investment strategies, long-term project planning, and policy analysis. The main technical result is a strongly sound and strongly complete logical system for coalition strategies with perfect recall. In addition to Sections 10, 11, and 12, an interesting possible direction of future work is to combine the proposed modality with temporal logic modalities.

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